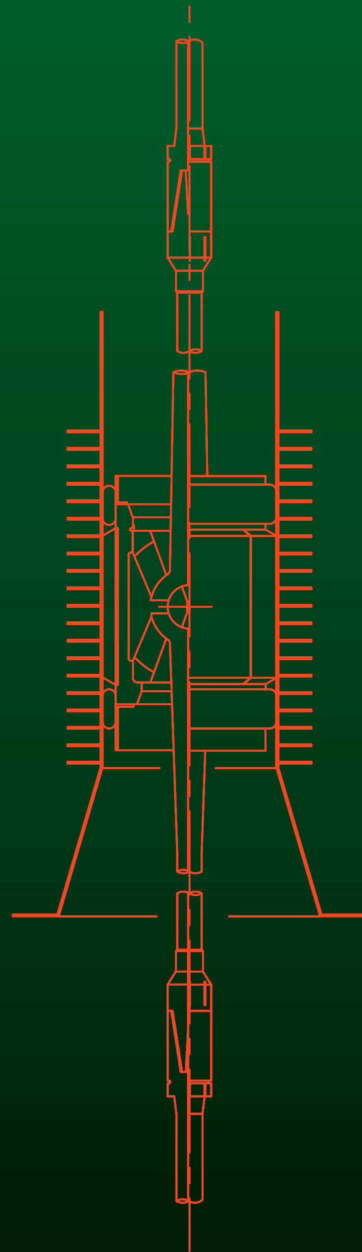
Design Engineer's Reference Guide

Mathematics, Mechanics, and Thermodynamics



Keith L. Richards

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Preface

In my career as a mechanical engineering designer, a recurring problem was finding information or a method of analysis for a specific design problem in which I was engaged at the time. In those early days, the Internet was not available, and substantial time was spent trying to obtain information. One of the first tasks I was given when entering the busy machine tool drawing office was to try and monitor the length of time designers and draughtsmen spent in obtaining design information. The surprising result was that a designer/draughtsman spent less than 10% of his or her time actually designing and drawing, and the remainder of the time was spent finding the relevant information. That led the company to invest in a technical library that included a wide selection of technical books and suppliers' catalogues. A follow-up survey approximately a year later showed that the library had indeed helped reduce the time spent finding relevant product and methods information.

In later life, when working in small design offices, I was often confronted with the same problems of finding relevant design information and as a result had to build up my own personal reference library. This book is a result of this work, and hopefully it will help other practising and student engineering designers who are faced with the same problem.

In Chapter 1, reference information is included covering such topics as trigonometry, differential and integral calculus, Laplace transforms, determinants and matrices.

In Chapter 2, numerical analysis, a broad subject matter, had to be reduced to covering numerical methods of integration, Newton–Raphson's methods, the Jacobi iterative method and the Gauss–Seidel method.

Chapter 3, 'Properties of Sections and Figures', is self-explanatory.

Chapter 4 covers statics, that is, forces in frameworks.

Chapter 5, although titled 'Dynamics', concentrates on kinematic analysis.

In Chapter 6, I have tried to cover the essentials of mechanical vibrations, including free, damped, simple harmonic and forced vibrations within the space available.

Chapter 7, 'Introduction to Control Systems', has been restricted to modelling individual elements of control systems. It is important to get the model correct before moving on to the analysis of the system, which is covered by other excellent books on the subject.

From my experience, most student engineers have difficulty understanding the use of transfer functions, and it was thought more important to concentrate on this aspect.

Chapter 8, heat and temperature, is a short chapter giving basic information on heat conduction and thermal expansions.

Chapters 9 and 10 cover the basics of thermodynamic and fluid dynamics.

Finally, Chapter 11 is an important one on the 'Introduction to Linkages'. The discussion is restricted to four-bar linkages, as these are the most common linkages that the student engineer will meet. It was felt that a discussion on six-bar linkages was outside the scope of this book and will be covered in a future publication on mechanism design.

At this point, I want to apologise for any mistakes or omissions; these are entirely my fault, and I would welcome any feedback on corrections among others so that they can be included in future reprints.

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Author

Keith L. Richards is a retired mechanical design engineer who has worked in the industry for over 55 years. Initially, he served an engineering apprenticeship with B.S.A. Tools, which manufactured a wide range of machine tools including the Acme Gridley, a multi-spindle automatic lathe built under licence, and the B.S.A. single spindle automatic lathes; these were used in Britain and widely exported around the world.

On leaving B.S.A., for a number of years he served as a freelance engineering designer in a wide range of industries, including aluminium rolling mill design, industrial forklift trucks and the Hutton tension leg platform, an offshore oil production platform. His responsibility on the latter project covered the mooring system components of the platform, and he answered to the customer (Conoco) and Lloyds inspectorate for all engineering aspects. In later years, Keith has been more involved in the aerospace industry, working on projects covering aircraft undercarriages and environmental control systems for military and commercial aircraft.

Acknowledgement

There are a number of wonderful people at Taylor & Francis/CRC Press I have to thank for bringing the manuscript into print; these include Jessica Vakili and Jonathan Plant for their help bringing this book into existence.

My wife deserves a very special thank you for all the help and support given whilst writing the script and to whom this book is dedicated.

1 Mathematics

1.1 TRIGONOMETRY

1.1.1 Right-Angled Triangle

(See Figure 1.1.)

$$\text{Area} = \frac{1}{2} a \cdot b \tag{1.1}$$

Pythagoras Theorem

$$c^2 = a^2 + b^2 \tag{1.2}$$

$$A + B + C = 180°, \quad B = 90° - A \tag{1.3}$$

$$\sin(90 - A) = \cos A, \quad \cos(90 - A) = \sin A \tag{1.4}$$

Definitions:

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} \tag{1.5}$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c} \tag{1.6}$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} \tag{1.7}$$

$$\text{cosecant } A = \frac{1}{\sin} = \frac{c}{a} \tag{1.8}$$

$$\text{secant } A = \frac{1}{\cos} = \frac{c}{b} \tag{1.9}$$

$$\text{cotangent } A = \frac{1}{\text{tangent}} = \frac{b}{a} \tag{1.10}$$

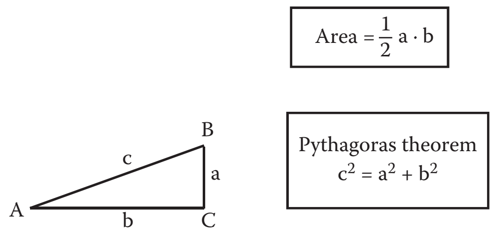


FIGURE 1.1 Right-angled triangle.

1.1.2 Oblique-Angled Triangles

(See Figure 1.2)

Cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A \tag{1.11}$$

$$b^2 = a^2 + c^2 - 2ac \cos B \tag{1.12}$$

$$c^2 = a^2 + b^2 - 2ab \cos C \tag{1.13}$$

Sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{1.14}$$

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2} ab \sin C \tag{1.15}$$

where

$$s = \frac{a+b+c}{2}$$

1.1.3 Trigonometric Relations

$$\sin(A)^2 = \frac{[1-\cos(2A)]}{2} \tag{1.16}$$

$$\sin(A)\cos(A) = \frac{\sin(2A)}{2} \tag{1.17}$$

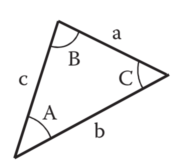


FIGURE 1.2 Oblique-angled triangle.

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) \tag{1.18}$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \tag{1.19}$$

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B) \tag{1.20}$$

$$\sin(A)\sin(B) = \frac{[1 - \cos(A+B) + \cos(A-B)]}{2} \tag{1.21}$$

$$\cos(-A) = \cos(A) \tag{1.22}$$

$$\sin(A)^2 + \cos A^2 = 1 \tag{1.23}$$

$$\cos(A)^2 = \frac{[1 + \cos(2A)]}{2} \tag{1.24}$$

$$1 + \cot(A)^2 = \operatorname{cosec}(A)^2 \tag{1.25}$$

1.2 HYPERBOLIC FUNCTIONS

Hyperbolic functions involve the exponential functions, e^x and e^{-x}, where e is the base of the Napierian logs (ln). (e = 2.7182828…).

Definitions:

$$\text{Hyperbolic sine:} \quad \sinh x = \frac{e^x - e^{-x}}{2} \tag{1.26}$$

$$\text{Hyperbolic cosine:} \quad \cosh x = \frac{e^x + e^{-x}}{2} \tag{1.27}$$

$$\text{Hyperbolic tangent:} \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x} \tag{1.28}$$

$$\cosh^2 x - \sinh^2 x = 1: \quad 1 - \tanh^2 x = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \tag{1.29}$$

(hyperbolic secant)

1.2.1 Inverse Hyperbolic Functions

Note: $\ln = \log_e$

$$y = \sinh^{-1}: \quad \text{'y' equals the inverse hyperbolic sinh of } x \tag{1.30}$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \quad \text{for all values of } x \tag{1.31}$$

$$\cosh^{-1} x = \pm\ln(x + \sqrt{x^2 - 1}) \quad x \geq 1 \tag{1.32}$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x} \quad x > 1 \tag{1.33}$$

1.3 SOLUTION OF THE QUADRATIC EQUATION

$$ax^2 + bx + c = 0 \quad a \neq 0, \text{and a, b and c are real} \tag{1.34}$$

The roots are

$$x = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac}) \tag{1.35}$$

$ax^2 + bx + c = 0$, $a \neq 0$ and a, b and c are real.

1.4 SOLUTION OF SIMULTANEOUS EQUATIONS (TWO UNKNOWNS)

$$a_{11}x_1 + a_{12}x_2 = b_1 \tag{1.36}$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \tag{1.37}$$

The solutions are

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \tag{1.38}$$

1.5 LAWS OF EXPONENTS

There are only six major laws of exponents that determine all that can be done with exponents and exponential functions:

1. $a^{x+y} = a^x a^y$ (1.39)
2. $a^{x-y} = \frac{a^x}{a^y}$ (1.40)
3. $(a^x)^y = a^{xy}$ (1.41)
4. $(ab)^x = a^x b^x$ (1.42)
5. $a^{x/y} = \sqrt[y]{a^x}$ (1.43)
6. $a^\circ = 1$ (1.44)

1.6 EXPANSIONS

$$(a \pm b)^2 = a^2 \pm 2ab + b^2 \tag{1.45}$$

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3 \tag{1.46}$$

$$(a^2 - b^2) = (a - b)(a + b) \tag{1.47}$$

$$(a^3 \pm b^3) = (a \pm b)(a^2 -/+ ab + b^2) \tag{1.48}$$

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots \tag{1.49}$$

1.7 REAL ROOT OF THE EQUATION F(X) = 0 USING THE NEWTON–RAPHSON METHOD

From Figure 1.3, it is seen that the derivative of y = f(x) is continuous and no point of inflexion exists between A and B.

x_0 is the true root (which is unknown) of f(x) = 0, that is when y = 0.
x_1 is a first approximation to x_0: obtained graphically or by trial and error.
A better approximation to x_0 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{1.50}$$

This assumes that x_1 is near enough to x_0.
This process may be repeated several times to obtain closer approximations to x_0.

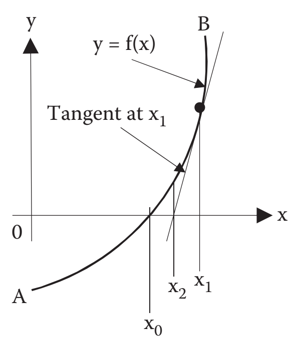


FIGURE 1.3 Real roots of the equation.

1.8 SERIES

Binomial:

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \tag{1.51}$$

Maclaurin:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \tag{1.52}$$

Taylor:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots \tag{1.53}$$

1.9 LOGARITHMS

If $y = x^n$ where n is the logarithm of y to the base x, that is, if $100 = 10^2$, 10 is the base and 2 is the logarithm.

10 is the base for *common* logarithms (log).

e = 2.71828… is the base for *natural* (Napierian) logarithms (ln).

$$\log(A \cdot B) = \log(A) + \log(B) \tag{1.54}$$

$$\log\left(\frac{A}{B}\right) = \log(A) - \log(B) \tag{1.55}$$

$$\log(A)^n = m\log(A) \tag{1.56}$$

Also

$$\ln(A \cdot B) = \ln(A) + \ln(B) \tag{1.57}$$

$$\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B) \tag{1.58}$$

$$\ln(A)^m = m\ln(A) \tag{1.59}$$

Note:

If $x = \log N$ or $x = \ln N$, then $N = 10^x$ or $N = e^x$.

N is the antilogarithm of x, that is, log (1) = 0 and log 10 = 1,

ln (1) = 0 and ln e = 1.

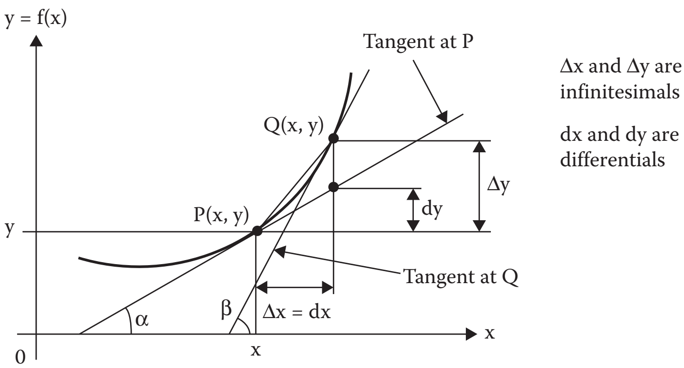


FIGURE 1.4 Definitions for differential calculus.

1.10 DIFFERENTIAL CALCULUS

Definitions:
(See Figure 1.4)

The value of the derivative at any point 'P' on a curve is equal to the slope of the tangent to the curve at that point.

$$\frac{\Delta y}{\Delta x} = \tan\beta \text{ (slope of tangent at P)} \tag{1.60}$$

as Δx becomes smaller, $\beta \to \alpha$,
that is

$$Q \to P \quad \frac{\Delta y}{\Delta x} \to \frac{dy}{dx} = \tan\alpha \tag{1.61}$$

Note:

$$\text{The first derivative } \frac{dy}{dx} \text{ is referred to as } y' = f'(x)$$

$$\text{and the second derivative } \frac{d^2y}{d^2x}, \quad y'' = f''(x), \text{ etc.}$$

Table 1.1 gives a range of standard derivatives for y or f(x).

1.11 INTEGRAL CALCULUS

Definitions:
(See Figure 1.5)

1.11.1 Integration Is the Inverse of Differentiation

If

$$F(x) = \int_a^b F(x)dx \quad \text{then } \frac{d}{dx}F(x) = F'(x) = f(x) \tag{1.62}$$

where F(x) is referred to as the primitive of the function f(x).

TABLE 1.1
Table of Standard Derivatives

y or f(x)	dy/dx or f′(x)
1. x	1
2. x^n	nx^{n-1}
3. $\frac{1}{x}$	$\frac{1}{x^{(2)}}$
4. $\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
5. a^x	$a^x \ln a$
6. ax^n	anx^{n-1}
7. e^x	e^x
8. eax	ae^{ax}
9. e^{nx}	ne^{nx}
10. $\log x$	$\frac{1}{x}$
11. $\ln ax$	$\frac{1}{x}$
12. x^x	$x^x(\log + 1)$
13. $\sin x$	$\cos x$
14. $\sin ax$	$a \cos ax$
15. $\cos x$	$-\sin x$
16. $\cos ax$	$-a \sin ax$
17. $\tan x$	$\sec^2 x$
18. $\tan ax$	$a \sec^2 ax$
19. $\cot x$	$-\mathrm{cosec}^2 x$
20. $\cot ax$	$-a\, \mathrm{cosec}^2 ax$
21. $\sec x$	$\sec x \cdot \tan x$
22. $\sec ax$	$a \sec ax \tan ax$
23. $\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cdot \cot x$
24. $\mathrm{cosec}\, ax$	$-a\, \mathrm{cosec}\, ax \cdot \cot ax$
25. $\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}} \quad \left(\lvert y \rvert < \frac{\pi}{2}\right)$
26. $\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2 - x^2}} \quad (0 < y < \pi)$
27. $\cos^{-1} x$	$-\frac{1}{\sqrt{(1-x^2)}} \quad (0 < y < \pi)$
28. $\cos^{-1} \frac{x}{a}$	$-\frac{1}{\sqrt{(a^2 - x^2)}}$
29. $\tan^{-1} x$	$\frac{1}{1+x^2}\left(\lvert y \rvert < \frac{\pi}{2}\right)$
30. $\tan^{-1} \frac{x}{a}$	$\frac{a}{(a^2+x^2)}$
31. $\sec^{-1} x$	$\frac{1}{x\sqrt{(x^2-1)}} \quad (0 < t < \pi x^2 > 1)$
32. $\sec^{-1} \frac{x}{a}$	$\frac{a}{x\sqrt{(x^2 - a^2)}}$

TABLE 1.1 (continued)
Table of Standard Derivatives

y or f(x)	dy/dx or f′(x)
33. $\text{cosec}^{-1} x$	$\frac{1}{x\sqrt{(x^2-1)}}$ $\left(\lvert y\rvert < \frac{\pi}{2} x^2 > 1\right)$
34. $\text{cosec}^{-1} \frac{x}{a}$	$\frac{1}{x\sqrt{(x^2-a^2)}}$
35. $\cot^{-1} x$	$\frac{1}{1+x^2}$ $\left(\lvert y\rvert < \frac{\pi}{2}\right)$
36. $\cot^{-1} \frac{x}{a}$	$-\frac{a}{a^2+x^2}$
37. $\sinh x$	$\cosh x$
38. $\sinh ax$	$a \cosh ax$
39. $\cosh x$	$\sinh x$
40. $\tanh x$	$\text{sech}^2 x$
41. $\tanh ax$	$a\,\text{sech}^2 ax$
42. $\coth x$	$-\text{cosech}^2 x$
43. $\coth ax$	$-\text{cosech}^2 ax$
44. $\text{sech } x$	$-\text{sech } x \cdot \tanh x$
45. $\text{sech } ax$	$-a\,\text{sech } ax$
46. $\text{cosech } x$	$-\text{cosech } x \cdot \coth x$
47. $\text{cosech } ax$	$-\text{cosech } ax \cdot \cosh ax$
48. $\sinh^{-1} x$	$\frac{1}{\sqrt{(1+x^2)}}$
49. $\sinh^{-1} \frac{a}{x}$	$\frac{1}{\sqrt{(a^2+x^2)}}$
50. $\cosh^{-1} x$	$\frac{1}{\sqrt{(x^2-1)}}$ $(y > 0, x^2 > 1)$
51. $\cosh^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{(x^2-a^2)}}$
52. $\tanh^{-1} x$	$\frac{1}{\sqrt{(1-x^2)}}$ $(x^2 < 1)$
53. $\tanh^{-1} \frac{x}{a}$	$\frac{a}{\sqrt{(a^2-x^2)}}$
54. $\coth^{-1} x$	$\frac{1}{(1-x^2)}$ $(x^2 > 1)$
55. $\coth^{-1} \frac{x}{a}$	$\frac{a}{(a^2-x^2)}$
56. $\text{sech}^{-1} x$	$-\frac{1}{\sqrt{(1-x^2)}}$ $(0 < x < 1)$
57. $\text{cosech}^{-1} x$	$-\frac{1}{x\sqrt{(x^2+1)}}$
58. $\text{cosech}^{-1} \frac{x}{a}$	$-\frac{a}{x\sqrt{(x^2+a^2)}}$

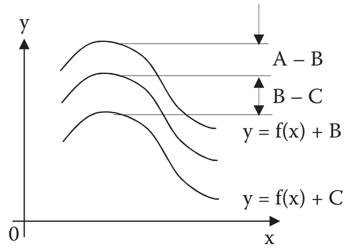


FIGURE 1.5 Definitions for integral calculus.

1.11.2 Indefinite Integrals

All integrals, $\int f(x)dx$ are classed as indefinite integrals and they differ from one another by a constant whose value depends upon the boundary conditions.

The following are some common integrals:

f(x)	**$\int f(x)dx$**
c (a constant)	cx
x^n	$x^{n+1}/(n+1)$ $(n \neq -1)$
1/x	ln x $(x \neq 0)$
e^{ax}	$e^{ax}/(a)$
ln x	x ln x – x (x > 0)
a^x	ax/(ln a) $(a > 0, a \neq 1)$
sin ax	–cos ax/(a)
cos ax	sin ax/(a)
tan x	–ln|sec x|

Note: The constants of integration have been omitted in the above table.

Table 1.2 gives a more comprehensive list of standard indefinite integrals.

1.11.3 Determination of an Area

Figure 1.6 depicts the area under a known curve f(x) that has been subdivided into vertical rectangular strips.

$$\text{Area } A = \lim_{\delta x_i \to 0} \sum_{i=1}^{i=n} f(x_i)\delta x_i = \int_a^b f(x)dx \quad (1.63)$$

The area is the summation of an infinite number of indefinitely small quantities:

$\int_a^b f(x)\,dx$ is the definite integral of the function f(x) between x = a and x = b

where a and b are the limits of integration.

TABLE 1.2
Table of Indefinite Integrals

f(x)	∫f(x) dx
1. $\int dx\ x$	$x + C$
2. $\int x^n\,dx$	$\frac{x^{n+1}}{n+1} + C \quad (n+1 \neq 0)$
3. $\int \frac{1}{x}dx$	$\ln\lvert x\rvert + C$
4. $\int \frac{1}{x \pm a\,dx}$	$\ln\lvert x \pm a\rvert + C$
5. $\int e^x\,dx$	$e^x + C$
6. $\int e^{nx}\,dx$	$\frac{1}{n}e^{nx} + C$
7. $\int a^x\,dx$	$\frac{a^x}{\ln a} + C \quad (a > 0, a \neq 0)$
8. $\int a^{nx}\,dx$	$\frac{a^{nx}}{n \cdot \ln a} \quad (a > 0, a \neq 0)$
9. $\int \ln x dx$	$x(\ln x - 1) + C$
10. $\int xe^{nx}\,dx$	$\frac{e^{nx}}{n^2}(nx - 1) + C$
11. $\int \sin x\,dx$	$-\cos x + C$
12. $\int \sin ax\,dx$	$-\frac{1}{a}\cos ax + C$
13. $\int \cos x\,dx$	$\sin x + C$
14. $\int \cos ax\,dx$	$\frac{1}{a}\sin ax + C$
15. $\int \tan x\,dx$	$-\ln\lvert\cos x\rvert + C$
16. $\int \cot x\,dx$	$\ln\lvert\sin x\rvert + C$
17. $\int \sin^2 x\,dx$	$\frac{1}{2}x - \frac{1}{4}\sin 2x + C$
18. $\int \cos 2x\,dx$	$\frac{1}{2}x + \frac{1}{4}\sin 2x + C$
19. $\int \sec 2ax\,dx$	$\frac{1}{a}\tan ax + C$
20. $\int \mathrm{cosec}\ 2ax\,dx$	$-\frac{1}{a}\cot ax + C$
21. $\int \frac{1}{\sin x\,dx}$	$\ln\tan\frac{x}{2} + C$
22. $\int \frac{1}{\cos x}dx$	$\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C$
23. $\int e^{nx}\sin bx\,dx$	$\frac{1}{n^2 + b^2}e^{nx}(n \cdot bx - b \cdot \cos bx) + C$
24. $\int e^{nx}\cos bx\,dx$	$\frac{1}{n^2 + b^2}e^{nx}(n \cdot \cos bx - b \cdot \sin bx) + C$
25. $\int \sin^{-1}x\,dx$	$x\sin^{-1}x + \sqrt{1 - x^2} + C$
26. $\int \cos^{-1}x\,dx$	$x\cos^{-1}x + \sqrt{1 - x^2} + C$
27. $\int \sinh x\,dx$	$\cosh x + C$
28. $\int \cosh x\,dx$	$\sinh x + C$

(continued)

TABLE 1.2 (continued)
Table of Indefinite Integrals

f(x)	∫f(x) dx
29. $\int \tanh x \, dx$	$\ln \cosh x + C$
30. $\int \frac{1}{\sqrt{a^2 - x^2}} dx$	$\sin^{-1} \frac{x}{a} + C \quad (\lvert x \rvert < a)$
31. $\int \frac{1}{\sqrt{a^2 + x^2}} dx$	$\frac{1}{a} \tan^{-1} \frac{x}{a} + C$
32. $\int \frac{1}{\sqrt{x^2 \pm a^2 dx}}$	$\ln\left(x + \sqrt{x^2 \pm a^2}\right) + C$
33. $\int \sqrt{a^2 - x^2} \, dx$	$\frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a}\right) + C$
34. $\int \frac{1}{x^2 - a^2} dx$	$\frac{1}{2a} \ln\left(\frac{x - a}{x + a}\right) + C \quad (x > a)$

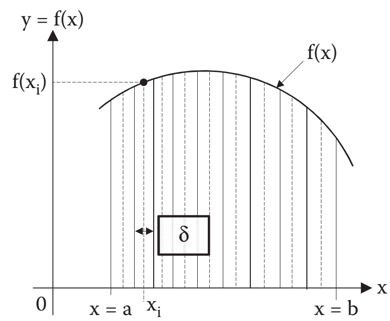


FIGURE 1.6 Determination of an area.

1.11.4 Approximate Integration

In an instance where the curve function is unknown, an approximate value of $\int_a^b f(x)dx$ can be found by two methods:

a. *Simpson's rule:* (See Figure 1.7)

$$\frac{h}{3}[y_1 + y_{n+1} + 4(y_2 + y_4 + y_6 + \cdots) + 2(y_3 + y_5 + y_7 + \cdots)] \tag{1.64}$$

where

$$h = (b - a)/n; \quad n = \text{even number of strips}$$

b. *Trapezoidal rule:*

$$h\left[\frac{1}{2}(y_1 + y_{n+1}) + y_2 + y_3 + \cdots + y_n\right] \tag{1.65}$$

In both methods, the accuracy can be improved by increasing the number of strips.

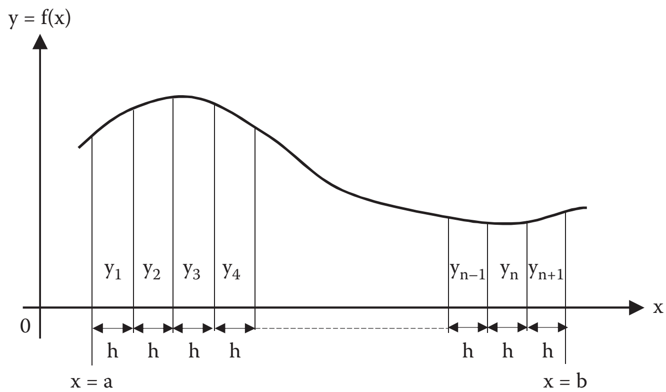


FIGURE 1.7 Method for approximation of an area using Simpson's rule.

Integration by parts:

$$\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx} v\, dx \tag{1.66}$$

Reduction formulae for trigonometric integrals:

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{m-1}{m}\int_0^{\pi/2} \sin^{m-2} x\, dx \tag{1.67}$$

$$\int_0^{\pi/2} \cos^m x\, dx = \frac{m-1}{m}\int_0^{p/2} \cos^{m-2} x\, dx \tag{1.68}$$

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2} x \cos^n x\, dx = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx \tag{1.69}$$

where n is an integer

(These results hold providing that the exponents in the reduced form are greater than –1. There are analogous reduction formulae with other intervals of integration ($1/2\ k_1\pi$, $1/2k_2\pi$) with k_1, k_2 integral.)

1.12 LAPLACE TRANSFORMS

The Laplace transform is a widely used integral transform used in many applications in engineering, particularly in vibration studies and control engineering. It is denoted by $\mathcal{L}\{f(t)\}$; it is a linear operator of a function f(t) with a real argument t ($t \geq 0$) that transforms f(t) to a function F(s) with complex argument 's'. There is a one-to-one correspondence in the transformation for the majority of practical uses; the most-common pairs of f(t) and F(s) are often given in tables for easy reference. The Laplace transform has the useful property that many relationships and operations over the original f(t) correspond to simpler relationships and operations over its image F(s).

It is named after Pierre-Simon Laplace, who introduced the transform in his work on probability theory.

Standard Laplace transforms are given in Table 1.3. The Laplace transforms of derivatives are as follows.

TABLE 1.3
Table of Standard Laplace Transforms

Function f(t)	Laplace Transforms $\mathcal{L}\{f(t)\} = \int_0^{*} e^{-st} f(t)\,dt$
1. 1	$\frac{1}{s}$
2. k	$\frac{k}{s}$
3. e^{at}	$\frac{1}{s-a}$
4. sin at	$\frac{a}{s^2+a^2}$
	$\mathcal{L}\left\{\frac{d^n y}{dx^n}\right\} = s^n \; L\{y\} - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{n-1}(0)$
5. cos at	$\frac{s}{s^2+a^2}$
6. t	$\frac{1}{s^2}$
7. t^2	$\frac{2!}{s^3}$
8. t^n (n = positive)	$\frac{n!}{s^{n+1}}$
9. cosh at	$\frac{s}{s^2-a^2}$
10. sinh at	$\frac{a}{s^2-a^2}$
11. $e^{-a}t\, t^n$	$\frac{n!}{(s+a)^{n+1}}$
12. e^{-at} sin ωt	$\frac{\omega!}{(s+a)^2+\omega^2}$
13. e^{-at} cos ωt	$\frac{s+a}{(s+a)^2+\omega^2}$
14. e^{-at} cosh ωt	$\frac{s+a}{(s+a)^2+\omega^2}$
15. e^{-at} sinh ωt	$\frac{\omega}{(s+a)^2-\omega^2}$

1.12.1 First Derivative

$$\mathcal{L}\left\{\frac{dy}{dx}\right\} = s\,\mathcal{L}\{y\} - y(0) \tag{1.70}$$

when y(0) is the value of y at x = 0.

1.12.2 Second Derivative

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} = s^2\;\mathcal{L}\{y\} - sy(0) - y'(0) \tag{1.71}$$

when $y'(0)$ is the value of $\frac{dy}{dx}$ at x = 0.

1.12.3 Higher Derivatives

$$\mathcal{L}\left\{\frac{d^n y}{dx^n}\right\} = s^n\ \mathcal{L}\{y\} - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{n-1}(0) \tag{1.72}$$

1.13 PARALLEL AXIS THEOREM

The parallel axis theorem may be used to refer the moment of inertia of a rigid body about a given axis to an offset parallel axis which is not necessarily the centre of mass of the body. The theorem is also known as the Huygens–Steiner theorem. The theorem has importance when calculating the sectional properties of a complex section.

1.13.1 Calculation of the Moment of Inertia Using the Parallel Axis Theorem

Consider the Figure 1.8 which shows a rectangular section offset to a specified axis. It is required to calculate the revised moment of inertia with respect to the new axis. In this simple case, the moment of inertia of the rectangle about its own centroid ‘x_c:x_c’ will be

$$I_c = \frac{b \cdot d^3}{12} \tag{1.73}$$

and the area will be

$$A_c = b \cdot d \tag{1.74}$$

The equation for the theorem is

$$I_{xx} = \sum (I_c + Ah^2) \tag{1.75}$$

EXAMPLE 1.1

Using a practical problem, consider the section shown in Figure 1.9. In this example, the section is broken down into simple rectangular sections. It is a simple matter to construct a table to evaluate the properties of these sections as shown in Table 1.4.

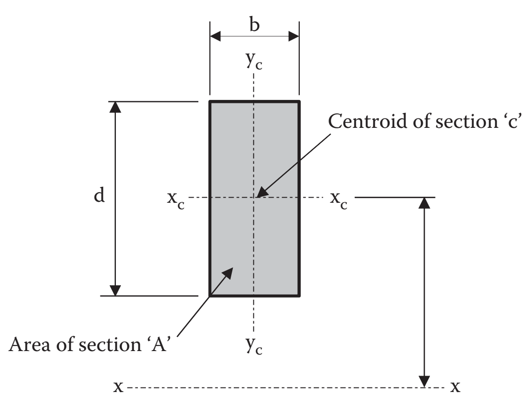


FIGURE 1.8 Parallel axis rule for area of moments.

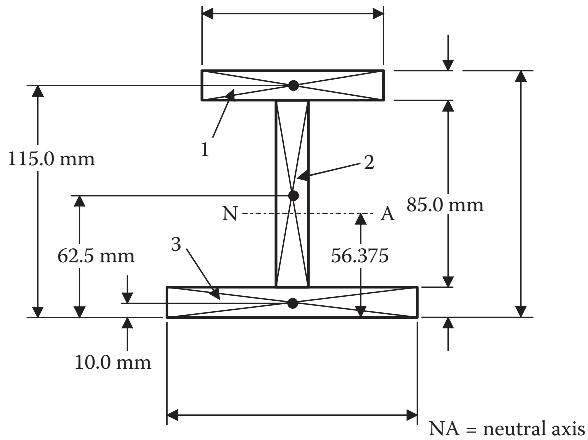


FIGURE 1.9 Example 'I' section.

From the table, it is seen that the individual moment of inertia is calculated about each section and the parallel axis theorem is then invoked to determine the new moment of inertia about the neutral axis.

The example shown is known as the *'Area Moment of Inertia'* as it deals with the respective areas of the section.

$$I_{\text{Parallel Axis}} = I_{\text{Centroidal Axis}} + Ah^2 \tag{1.76}$$

where

A is the cross-sectional area.

h is the perpendicular distance between the centroidal axis and the parallel axis.

TABLE 1.4
Sectional Properties for Example 1.1

Section Number	b (mm)	d (mm)	A (mm^2)	y (mm)	A·y (mm^3)
					$\times 10^3$
1	90.0	20.0	1800	115.0	207,000
2	20.0	85.0	1700	62.5	106,250
3	125.0	20.0	2500	10.0	25,000
			$\Sigma A = 6000\ mm^2$		$\Sigma Ay = 338{,}250\ mm^3$

Position of neutral axis: $\bar{Y} = 56.375$ mm

Area Moment of Inertia Calculations

Section Number	I (mm^4)	h (mm)	$I_{xx} + (A \cdot h^2)$ (mm^4)
			$\times 10^3$
1	3000.0	58.6	6189.403
2	12,041.7	6.1	75.818
3	4166.7	46.4	5380.768
	$\Sigma I = 19{,}208.3\ mm^4$		$\Sigma I_{xx} = 11{,}645.990\ mm^4$

The theorem can also be used to determine the *'Mass Moments of Inertia'*

$$I = Mh^2 \tag{1.77}$$

where
M is the mass of the body.
h is the perpendicular distance between the centroidal axis and the parallel axis.

The *'Mass Radius of Gyration'* of a section is a measure of the distribution of the mass in an object from its geometric centre and is a sign of its resistance to rotational motion.
The radius of gyration can also be referred to as an offset parallel axis in a similar manner.

$$k^2 = k_c^2 + h^2 \tag{1.78}$$

where
k is the radius of gyration about an axis parallel to the centroidal axis.
$k_{c'}$ is the radius of gyration about the centroidal axis.
h is the perpendicular distance between the centroidal axis and the parallel axis.

1.14 COMPLEX NUMBERS

1.14.1 Introduction

When finding the solution to a quadratic equation such as

$$ax^2 + bx + c = 0 \tag{1.79}$$

there will always be two solutions. In most cases, the solution will be straightforward, but in the case of the equation

$$x = 5x^2 - 6x + 5 = 0 \tag{1.80}$$

using the standard formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} \tag{1.81}$$

the solution will be

$$x = \frac{6 \pm \sqrt{(-64)}}{10} \tag{1.82}$$

The solution to this equation relies on solving $\sqrt{}(-64)$.
$\sqrt{}(-64)$ cannot be represented by an ordinary number as there is no real number whose square is a negative quantity.
Now −64 can be written as -1×64, and from this:

$$j^7 = j^4 \times j^3 = -j$$

$$= \sqrt{(-1)} \times \sqrt{(64)}$$

$$= 8\sqrt{(-1)} \tag{1.83}$$

This still leaves one to find the solution of √(–1). This cannot be evaluated as a real number, but if √(–1) is represented by j, this will make the workings a lot easier.

The final solution to the quadratic therefore will be

$$x = \frac{6 \pm j8}{10} \quad \therefore x = 0.6 \pm j0.8 \tag{1.84}$$

$$\therefore x = 0.6 + j0.8 \quad \text{or} \quad x = 0.6 - j0.8 \tag{1.85}$$

Now

$$j = \sqrt{-1} \tag{1.86}$$

$$\therefore j^2 = \sqrt{-1} \times \sqrt{-1} = -1 \tag{1.87}$$

$$j^3 = j^2 \times j = -j \tag{1.88}$$

$$j^4 = j^2 \times j^2 = 1 \tag{1.89}$$

$$j^5 = j^4 \times j^2 = j \tag{1.90}$$

$$j^7 = j^4 \times j^3 = -j \tag{1.91}$$

1.14.2 Argand Diagram

The Argand diagram was devised as a means of representing complex numbers.

Although Jean Robert Argand, a Swiss mathematician in 1806, is credited with developing the diagram, it was in fact described by C. Wessel earlier. The geometric representation of complex numbers as points in a plane made the whole idea of a complex number more acceptable. Indeed, this visualisation helped 'imaginary' and 'complex' numbers become more accepted in mainstream mathematics as a natural extension to negative numbers along the real line.

Figure 1.10 shows the diagram with the complex number $z = a + jb$ plotted as a point with coordinates (a, b). Because the real part of z is plotted on the horizontal axis of the diagram, this is often referred to as the 'real axis'. The imaginary part of z is plotted on the vertical axis and this is then referred to as the 'imaginary axis'.

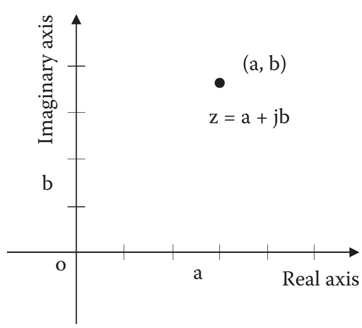


FIGURE 1.10 Argand diagram.

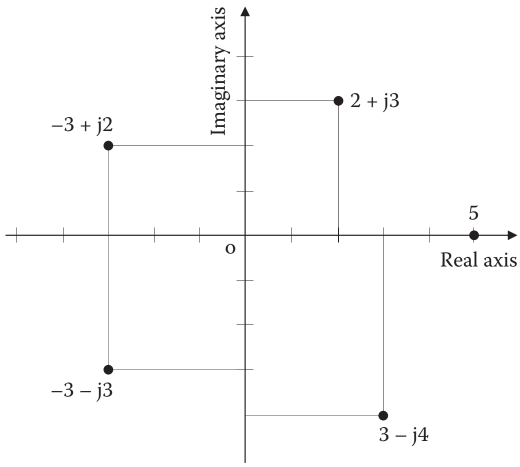


FIGURE 1.11 Argand diagram for Example 1.2.

EXAMPLE 1.2

Plot the complex numbers 2 + j3, –3 + j2, –3 – j3, 3 – j4 and 5j on an Argand diagram.

Solution

This is shown in Figure 1.11.

1.14.3 Manipulation of Complex Numbers

1.14.3.1 Addition and Subtraction

The real parts and the imaginary parts of the numbers are added or subtracted separately, for example,

$$(3 + j4) + (5 + j2) = 8 + 6j \tag{1.92}$$

1.14.3.2 Multiplication

Multiplication is undertaken algebraically; complex multiplication is more difficult to understand from either an algebraic or geometric point of view. Carrying out the algebraic operation first,

$$(3 + j2) \quad \text{and} \quad (1 + j4) \tag{1.93}$$

each of these have two terms each and when multiplied together will result in four terms.

$$(3 + j2)(1 + j4) = 3 + 1j2 + j2 + j28 \tag{1.94}$$

The 1j2 + j2 terms will reduce to j14, which leaves the $j^2 8$ term. Remember that ‘j’ was introduced as an abbreviation for √(–1), the square root of –1. Therefore, ‘j’ is something whose square is –1 (see Equation 1.87).

Thus, j28 will equal –8. The product of

$$(3 + j2)(1 + j4) = (-5 + j14)$$

From this example, it is possible to arrive at a general rule for the solution of multiplying complex numbers.

$$(x + jy)(u + jv) = (xu - yv) + j(xv + yu)$$

Remember that (xu – yv) is the real part of the product and is the product of real parts minus the products of the imaginary parts. But (xv + yu), the imaginary part of the product, is the sum of the two products of one real part and the other imaginary part.

1.14.3.3 Division

Consider

$$\frac{2 + j3}{3 - j4} \tag{1.95}$$

Now

$$\frac{2 + j3}{3 - j4} = \frac{2 + j3}{3 - j4} \times \frac{3 + j4}{3 + j4} = \frac{(2 + j3)(3 + j4)}{(3^2 - 4^2)} \tag{1.96}$$

$$= \frac{6 + j8 + j9 - 12}{25}$$

$$= \frac{-6 + j17}{25}$$

$$= \frac{-6}{25} + \frac{j17}{25}$$

$$= 0.24 + j0.68 \tag{1.97}$$

1.14.4 Polar Form of a Complex Number

It is sometimes convenient to express a complex number (a + jb) in a different form. From an Argand diagram, OP is a vector a + jb. Let r = length of the vector and θ the angle made with the axis OX (Figure 1.12).

Now

$$r^2 = a^2 + b^2 \quad r = \sqrt{(a^2 + b^2)} \tag{1.98}$$

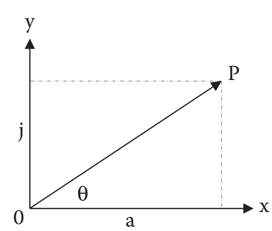


FIGURE 1.12 Polar form of a complex number.

And

$$\tan\theta = \frac{b}{a}$$

Also

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta \tag{1.99}$$

Since z = a + jb, this can be rewritten

$$z = r\cos\theta + jr\sin\theta \quad \text{that is, } z = r(\cos\theta + j\sin\theta) \tag{1.100}$$

This is called the 'Polar Form' of the complex number a + jb, where

$$r = \sqrt{(a^2 + b^2)} \quad \text{and} \quad \theta = \tan^{-1}\frac{b}{a}$$

EXAMPLE 1.3

Express z = 4 + j3 in polar form.

Solution

From Figure 1.13, it can be seen that

a. $r^2 = 4^2 + 3^2 = 16 + 9$
 $= 25$ therefore, $r = 5$.

b. $\tan\theta = \frac{3}{4}$
 $= 0.75$
 $\theta = 36.8667°$
 $z = a + jb = r(\cos\theta + j\sin\theta)$

Therefore, in this case

$$z = 5(\cos 36.8667° + j\sin 36.8667°)$$

EXAMPLE 1.4

Find the polar form of the complex number 2 + j3.

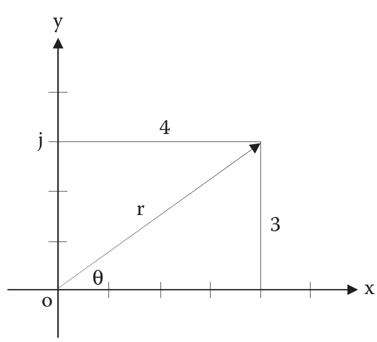


FIGURE 1.13 Argand diagram for Example 1.3.

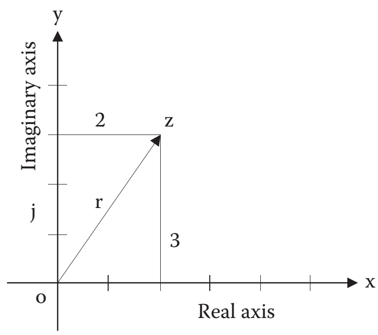


FIGURE 1.14 Argand diagram for Example 1.4.

Solution

First, construct a sketch diagram (see Figure 1.14).

$z = 2 + j3 = r(\cos\theta + j\sin\theta)$
$r^2 = 4 + 9 = 13$
$r = 3.6056$
$\tan\theta = \dfrac{3}{2}$
$= 56.310°$
$z = 3.6056\,(\cos 56.310° + j\sin 56.310°)$.

1.14.5 Exponential Form of a Complex Number

Thus far, two ways of representing a complex number have been considered, namely:

- Cartesian form ($z = a + jb$) (1.101)
- Polar form ($z = r[\cos\theta + j\sin\theta]$) (1.102)

In this section, a third method of denoting a complex number is introduced:

- Exponential form ($z = r \cdot e^{j\theta}$) (1.103)
  The exponential form is derived from the polar form; therefore:
  - The value of 'r' is the same in both cases.
  - The angle is also the same in each case (in the exponential form, the angle has to be in radians).

EXAMPLE 1.5

Convert the polar form $7(\cos 60° + j\sin 60°)$ into the exponential form.

Solution

Converting to exponential form $7(\cos 60° + j\sin 60°)$:

'r' = 7
'θ' = 60° (in radian form = $\dfrac{\pi}{3}$ radians).
Therefore, in exponential form: $= 7e^{j\pi/3}$.

If θ is replaced by −θ, the following will result:

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta)$$

$$= \cos\theta - j\sin\theta$$

Summarising:

$$e^{j\theta} = (\cos\theta + j\sin\theta)$$

$$e^{-j\theta} = (\cos\theta - j\sin\theta)$$

1.15 DETERMINATES

1.15.1 Introduction

A determinant is an algebraic operation in which a square matrix is reduced to a numerical value (scalar). It has many uses, particularly in the inversion of matrices.

The notion of determinants predates matrices and linear transformations. Cardano considered determinants towards the end of the sixteenth century and Leibniz also studied them approximately 100 years later.

1.15.2 Description

A determinant is a square array of quantities (elements) to which a numerical value is assigned. Their use is in the analysis and solutions of systems of linear algebraic solutions. In general, the solutions are unmanageable when written out in length. The use of determinants helps simplify the expressions.

As an example of reducing linear algebraic equations to a determinant, consider the following simultaneous equation:

$$a_{11}x_1 + a_{12}x_2 + a_{13} = 0 \quad (1.104)$$

$$a_{22}x_1 + a_{22}x_2 + a_{23} = 0 \quad (1.105)$$

The solution is obtained in the form

$$\frac{x_1}{(a_{12}\cdot a_{23} - a_{22}\cdot a_{13})} = \frac{x_2}{(a_{13}\cdot a_{22} - a_{23}\cdot a_{11})} = \frac{1}{(a_{11}\cdot a_{12} - a_{22}\cdot a_{12})} \quad (1.106)$$

The denominators of this solution can then be expressed in the form of determinants:

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}\cdot a_{23} - a_{22}\cdot a_{13} \quad (1.107)$$

$$\begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{22} \end{vmatrix} = a_{13}\cdot a_{22} - a_{23}\cdot a_{11} \quad (1.108)$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\cdot a_{22} - a_{21}\cdot a_{12} \quad (1.109)$$

1.15.3 Determinant Order

The number of rows/columns in a square matrix determines the order of the determinant, that is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \text{Second-order determinant} \tag{1.110}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \text{Third-order determinant} \tag{1.111}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \text{Fourth-order determinant} \tag{1.112}$$

Consider a third-order determinant, that is, three rows and three columns:

$$\text{Determinant} = \text{Det}\,|A| = D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{1}^{3} a_{ij} \tag{1.113}$$

The value of the determinant

$$D = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \tag{1.114}$$

where the signs are assigned in accordance with the following table:

$$\begin{vmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \end{vmatrix}$$

Signs are associated with determinant element positions.

From the above, it is seen that elements are assigned either positive or negative signs depending on their position within the determinant.

1.15.4 Properties of the Determinant

1. The value of the determinant remains unchanged if rows and columns are interchanged.
2. If two rows of the determinant are interchanged, the numerical value of the determinant is unaltered but its sign is changed.
3. If two rows (or columns) of the determinant are identical, then the value of the determinant is zero.
4. If all the elements of one row (or one column) of the determinant are multiplied by 'k', the new determinant equals $k\Delta$.

1.15.5 Minors and Cofactors

Minor of a determinant element: Consider the fourth-order determinant shown in Equation 1.112. A 'minor' is the determinant of the square matrix formed by deleting one row and one column from some larger square matrix in which the determinant element is required.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

The minors are calculated as follows:

$$\text{The minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}; \quad \text{The minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$\text{The minor of } a_{14} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

This procedure is then repeated for the remaining elements in the determinant and the cofactor of the determinant is then made up of a matrix representing the minors of the elements. Numerical values are assigned to the elements and the cofactor will be a single numerical value for the determinant.

1.16 MATRICES

1.16.1 Introduction

A matrix is an array of mathematical quantities that is used in the solution of linear algebraic equations. Matrices are important in engineering, statistics, physics and so on.

The following is an example of a linear algebraic equation expressed in a matrix form

$$\underbrace{\begin{matrix} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{matrix}}_{\text{Linear algebraic equation}} = \underbrace{\underset{\text{Column}}{\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}} = \underset{\text{Square}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{\text{Column}}{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}}}_{\text{Matrices}} \underset{\text{Short-hand notation}}{\equiv \{y\} = [A]\{x\}}$$

From the above, it is seen that (x) and (y) are column matrices (these are often referred to as vectors) and (a) is a square matrix.

1.16.2 Definitions

The following are the basic definitions that are used in matrix algebra.

1.16.2.1 Square Matrix

A square matrix consists of (n) rows and (m) columns and is referred to as a matrix of order $n \times m$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix} = [A] \tag{1.115}$$

The matrix equation can also be written as

$$y_i = \sum_{i=1}^{n} a_{ij} x_i$$

where i = 1,2,3; i is the number of rows and j the number of columns.

1.16.2.2 Row Matrix

A row matrix has one row of numbers as shown below

$$[a_1 \quad a_2 \quad a_3 \ \ldots \ a_n] = [A]$$

1.16.2.3 Column Matrix

A column matrix consists of a single column of numbers as shown below

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [A]$$

1.16.2.4 Diagonal Matrix

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

1.16.2.5 Unit Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.16.2.6 Symmetric Matrix

$$a_{ij} = a_{ji} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

1.16.2.7 Skew Symmetric Matrix—That Is, Anti-Symmetric ($a_{ij} = -a_{ji}$)

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

1.16.2.8 Null Matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where all the elements are zero.

1.16.3 Matrix Algebra

1.16.3.1 Additions of Matrices

Addition of matrices is completed as follows:

$$[A] + [B] = [a_{ij} + b_{ij}] = \begin{bmatrix} (a_{11} + b_{11}) & (a_{12} + b_{12}) & \ldots \\ . & . & . \\ . & . & . \\ . & . & . \\ . & . & (a_{mn} + b_{mn}) \end{bmatrix}$$

1.16.3.2 Multiplication of Matrices

Multiplication of matrices is completed as follows:

$$\begin{array}{cc} & \begin{bmatrix} \text{Square} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix} \Leftarrow [B] \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} & \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix} \\ \Uparrow & \Uparrow \\ [A] & [C] = [A][B] \end{array}$$

For example, $c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$.

In general, $[A][B] \neq [B][A]$.

1.16.3.3 Transposition of a Matrix

Transposition of a matrix is where each row element becomes a column element and vice versa:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{Transpose: } [A]^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

1.16.3.4 Adjoint of a Matrix

Adjoint of a matrix is where the cofactors (c_{ij}) are transposed:

$$[\text{Adj } A] = [C]^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

For a definition of cofactors, see Section 1.15.4 on determinants.

1.16.3.5 Inverse of a Square Matrix

$$[A] = [A]^{-1} = \frac{[A]^T}{\text{Det}[A]} = \frac{[A]^T}{|a|}$$

The inverse can only be defined for a square matrix. (Note: There are cases where a square matrix cannot be defined.)

1.16.3.6 Transformation from Cylindrical Coordinates to Cartesian Coordinates

An example of the use of matrices is the transformation of cylindrical to Cartesian coordinates in vector algebra. A transformation vector can be determined [A] such that the Cartesian coordinate $\{V\}_{cart} = [A]\{V\}_{cyl}$.

$$[A] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_R \\ V_0 \\ V_Z \end{bmatrix}$$

$$\{V\}_{cart} = [A]\{V\}_{cyl}$$

2 Introduction to Numerical Methods

2.1 INTRODUCTION

Before the advent of electronic digital computers, all computing was accomplished using manual methods.

The aim of this introduction is to present some of the methods used in analysing practical problems arising in mechanical engineering that are not generally solvable using classical methods. These methods were developed prior to the development of the digital computer and have stood the test of time.

The basic mathematics used in numerical analysis is covered in Chapter 1.

The analysis of a physical problem will involve four basic steps:

1. Development of a suitable mathematical model that realistically represents the physical system
2. Derivation of the systems governing equations
3. Solution of the governing equations
4. Interpretation of the results

While an analytical solution using classical methods will be exact, if it exists, a numerical method will require a number of iterations to generate a solution; this is only an approximation and cannot be considered to be exact by any means.

It is important to understand the errors that arise in numerical analysis, and for this reason a separate section devoted to errors is covered in Section 2.2 and covers the difference between round-off errors and truncation errors.

The numerical methods will be able to solve most complex problems, and one of the advantages is easy programming on a computer using programs such as MATLAB®, MathCAD, Maple and Mathematica. All of these programs are user-friendly.

2.2 NUMERICAL METHODS FOR INTEGRATION

There may be a point where the engineer is required to calculate a definite integral that cannot be solved using analytical integration or it is preferred to integrate tabulated data.

The following methods may be used to solve for a definite integral:

1. Manual method
2. Mid-ordinate rule
3. Trapezoidal rule
4. Simpson's rule
5. Quadratic triangulation
6. Romberg integration
7. Gauss quadratic

In this section, the first four methods will be considered as being the most appropriate ones for engineering students to understand.

2.2.1 Manual Method

This method is about the simplest to implement, where the desired curve is traced onto a piece of graph paper. The squares under the curve are counted where they cover more than 50% of the function as in Figure 2.1. The result may give a reasonable estimate of the area under the curve, provided the grid is sufficiently fine.

2.2.2 Mid-Ordinate Rule

The mid-ordinate rule is also referred to as a quadrature or rectangular rule.

In this method, the area to be integrated is sub-divided into rectangles of equal width. The strips may be an even or odd number. In Figure 2.2, the value of $A = \int_a^b f(x)dx$ represents the area under the curve between the values x = a and x = b. As stated above, the area is divided into strips of equal width, w = (b − a)/n, where n is the number of strips. The height of each rectangle is measured at the point where the mid-point of the strip crosses the curve, that is, halfway between x = 0 and x = 1, that is x = 0.5. The height of the second rectangle will correspond to the 'y'-value for x halfway between x_1 and x_2—namely, at a height of $y_{1.5}$.

This procedure is continued until the mean of the final x-values, x_{n-1} and x_n, is reached and the final y-value, $y_{n-0.5}$.

The formula can be expressed as

$$A = h(y_{0.5} + y_{1.5} + y_{2.5} + y_{n-0.5}) \tag{2.1}$$

where

$$h = \frac{b - a}{n}$$

Purely as an exercise for those readers unfamiliar with the method, it is required to integrate the curve $y = x^2$ between the values of x = 3 and x = 9. Starting with four rectangular strips, each with a width of four units as in Figure 2.1, it is noted that each strip crosses the curve at its mid-point and

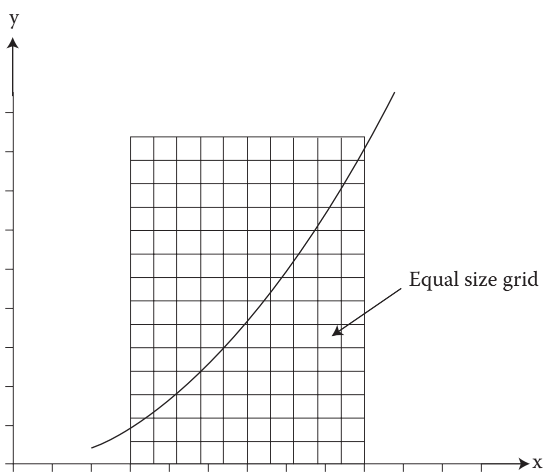


FIGURE 2.1 Manual method of determining the area under a curve.

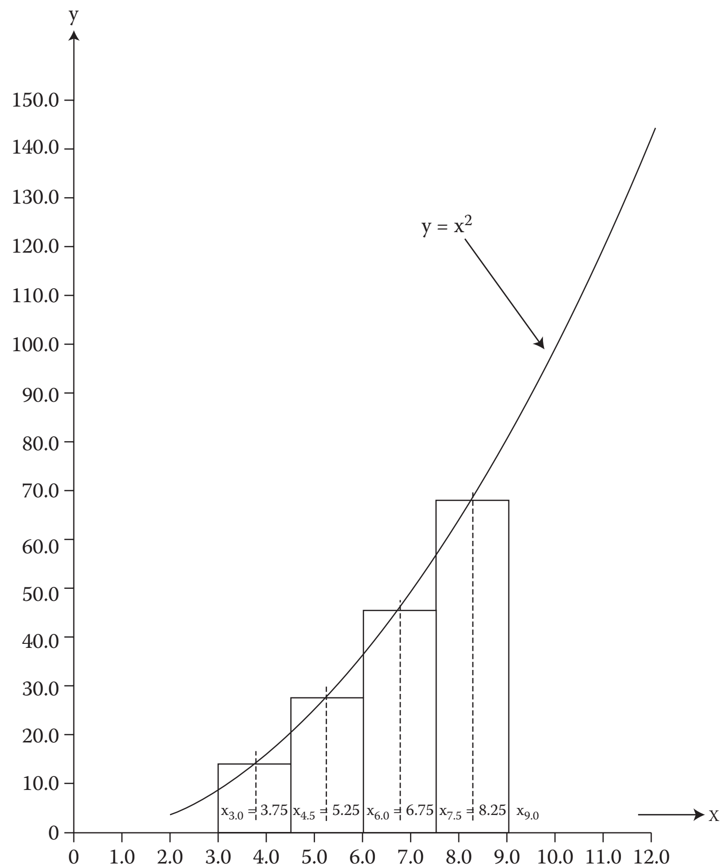


FIGURE 2.2 Mid-ordinate rule.

the area of the strip is then computed by multiplying the length of the strip by its width. The integral is the sum of the total areas of the strips. Cheating a little, it is already known that the integral uses analytical methods:

$$y = \int_3^9 x^2 \, . dx$$

$$= \left[\frac{x^3}{3}\right]^9$$

$$= \left[\frac{9^3}{3}\right] - \left[\frac{3^3}{3}\right]$$

$$= 234 \text{ units}$$

Returning to Figure 2.2, tabulating the results of the measured areas, Table 2.1 is constructed.

$$\text{Total area} = x_1(y_1 + y_2 + y_3 + y_4) \ \ (\text{where } x_1 = x_2 = x_3 = x_4) = 238.35 \text{ mm}^2 \qquad (2.2)$$

Increasing the number of strips will increase the accuracy of the integrand.

TABLE 2.1
Strip Dimensions for Figure 2.2

Number of blocks	4
Width of strip	1.5 mm
Height of strips	
y_1	14.52 mm
y_2	27.90 mm
y_3	45.90 mm
y_4	68.40 mm

TABLE 2.2
Effect on Values of Integrand with the Increase in Number of Strips

Number of Strips	Integral Value
4	238.35
6	237.12
8	236.62
12	236.36
24	236.19

Table 2.2 tabulates the results of a series of calculations using an increased number of segment widths. It is clearly seen that as the segment strip width is reduced, the values converge.

2.2.3 Trapezoidal Rule

A variant of the mid-ordinate rule is the trapezoidal rule which more closely approximates the curve more accurately than the mid-ordinate rule (Figure 2.3). As with the mid-ordinate rule, the number of strips used is immaterial; they may be an even or odd number.

The strip is made up of an element containing both a rectangle and a triangle as shown in Figure 2.4a and b. The area of the trapezoid is obtained by adding the two areas together.

$$A = y_o\Delta x + \frac{1}{2}(y_1 - y_o)\Delta x = \frac{(y_o + y_1)\Delta x}{2} \tag{2.3}$$

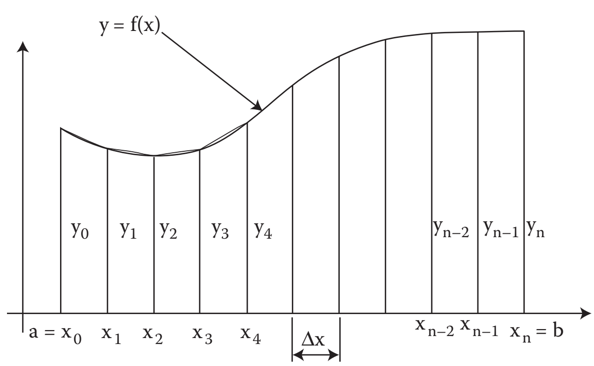


FIGURE 2.3 Trapezoidal rule.

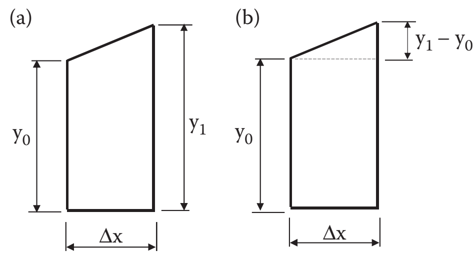


FIGURE 2.4 (a) and (b): Trapezoidal elements.

Adding the area of n trapezoids, the following approximation is obtained:

$$\int_a^b f(x)dx \approx \frac{(y_o + y_1)\Delta x}{2} + \frac{(y_1 + y_2)\Delta x}{2} + \frac{(y_2 + y_3)\Delta x}{2} + \cdots + \frac{(y_{n-1} + y_n)\Delta x}{2} \tag{2.4}$$

This simplifies to the trapezoidal rule formula

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(y_o + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \tag{2.5}$$

EXAMPLE 2.1

Using the trapezoidal rule, estimate the following expression with n = 8.

$$\int_1^5 \sqrt{1 + x^2}\ dx$$

Solution

For n = 8, the strip size Δx will be

$$\Delta x = \frac{5 - 1}{8} = 0.5$$

The values of $y_o, y_1, y_2, \ldots, y_8$ can now be computed as shown in Table 2.3. Therefore,

$$\int_1^5 \sqrt{1 + x^2}\ dx \approx \frac{0.5}{2}\Big(\sqrt{2} + 2\sqrt{3.25} + 2\sqrt{5} + 2\sqrt{7.25} + 2\sqrt{10}$$
$$+ 2\sqrt{13.25} + 2\sqrt{17} + 2\sqrt{21.25} + \sqrt{26}\Big)$$

which resolves to

$$\approx 12.76 \text{ (Ans)}$$

TABLE 2.3
Measured Values Used in Example 2.1

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$y = \sqrt{1+x^2}$	$\sqrt{2}$	$\sqrt{3.25}$	$\sqrt{5}$	$\sqrt{7.25}$	$\sqrt{10}$	$\sqrt{13.25}$	$\sqrt{17}$	$\sqrt{21.25}$	$\sqrt{26}$

TABLE 2.4
Measured Values Used in Example 2.2

x	2.1	2.4	2.7	3.0	3.3	3.6
y	3.2	2.7	2.9	3.5	4.1	5.2

EXAMPLE 2.2

The results shown in Table 2.4 were measured in an experiment.

Using the trapezoidal rule, estimate $\int_{2.1}^{3.6} y\,dx$.

Solution

By inspection, dx = 0.3.

Therefore,

$$\int_{2.1}^{3.6} y\,dx \approx \frac{0.3}{2}(3.2 + 2(2.7) + 2(2.9) + 2(3.5) + 2(4.1) + 5.2)$$

$$\approx 5.22 \text{ (Ans)}$$

In the case of the curve in Figure 2.1, the results from the trapezoidal rule will be identical to the mid-ordinate rule. The trapezoidal rule has the advantage where the curve is not continuous as seen in Figure 2.3; with the strip being made up of a rectangle and a triangle, it follows the curve more accurately and reduces the error even with larger strip widths.

2.2.4 Simpson's Rule

Where the trapezoidal rule is more accurate than using the mid-ordinate rule, an even better approximation can be made using Simpson's rule where parabolas are used instead of straight lines to approximate each part of the curve. Simpson's rule is widely used by engineers in preference to most other methods of integration (see Figure 2.5).

As with the previous methods, the area under the required curve is partitioned into an even number of sub-intervals having equal widths such that:

$$\{x_0, x_1, x_2, x_3, x_4, \ldots, x_{n+1}\} \tag{2.6}$$

where n is an even number of strips.

The width of each sub-interval is given by (see Figure 2.5)

$$\Delta x = \frac{b-a}{n}$$

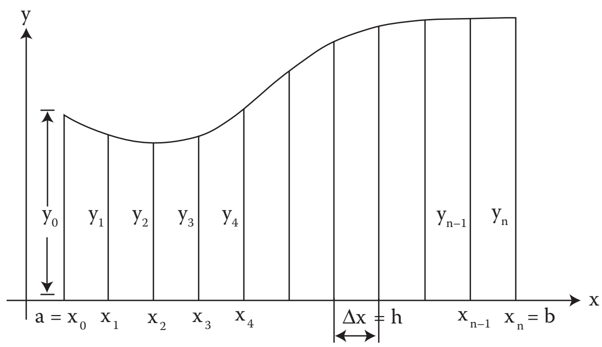


FIGURE 2.5 Simpson's rule.

The area under the curve is given by the following formula

$$A \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \cdots + 4y_{n-1} + y_n) \tag{2.7}$$

By way of an example to the method, consider the definite integral.

Approximate $\int_3^3 [dx/(x+1)]$ using Simpson's rule with n = 4.

$$\Delta x = \frac{b-a}{4}$$

$$= 0.25$$

$$y_0 = f(a) = f(2) = 0.333333$$

$$y_1 = f(a + \Delta x) = f(2.25) = \frac{1}{2.25+1} = 0.307692$$

$$y_2 = f(a + 2\Delta x) = f(2.5) = \frac{1}{2.5+1} = 0.285714$$

$$y_3 = f(a + 3\Delta x) = f(2.75) = \frac{1}{2.75+1} = 0.266667$$

$$y_4 = f(b) = f(3) = \frac{1}{3+1} = 0.25$$

Hence,

$$\text{Area} = \int_a^b (x)dx$$

$$\approx \frac{0.25}{3}[0.333333 + 4(0.307692) + 2(0.285714) + 4(0.266667) + 0.25]$$

$$= 0.287683$$

The actual answer is 0.287682; therefore, the rule is in error by only 0.00036%.

2.3 EVALUATION OF ERRORS

From the foregoing section on solving for a definite integral, it was seen how errors arising from the calculation can be reduced by the choice of method, that is, mid-ordinate, triangulation or Simpson's rule, and increasing the number of strips used in the calculation.

2.4 ROUND-OFF AND TRUNCATION ERRORS

Errors arising from the solution of engineering problems can occur due to several factors. The error may be due to

1. Incorrect assumption in the modelling technique, that is that drag force is proportional to the velocity of a vehicle when it is actually proportional to the square of the velocity
2. Errors from mistakes in the programs themselves
3. Errors in the measurement of the physical properties
4. Mixing imperial units with SI units or using incorrect conversion values

In the application of numerical methods, there are two types of error that need to be considered; these are

1. Round-off errors
2. Truncation errors
3. Errors arising from differentiation
4. Integration errors

2.4.1 Round-Off Errors

Round-off errors depend upon the level of precision the engineer is working to.

Consider the number $\frac{1}{4}$. This is represented as 0.25 in decimal format, which is an exact value, but a number such as $\frac{1}{3}$ would be represented as 0.333333 recurring. The round-off error in this case will be $\frac{1}{3} - 0.333333 = 0.0000003\overline{3}$ (the over-bar represents that the last digit is recurring).

There are other numbers that cannot be represented exactly, such as π or $\sqrt{2}$; these are known as transcendental numbers. They continue to infinity without repeating and therefore need to be approximated in calculations.

When there are calculations involving a number of decimal values that have been either rounded up or down, then the errors in the final calculation can become significant, and these errors need to be considered carefully.

2.4.2 Truncation Errors

A truncation error is defined as an error occurring when a mathematical procedure is prematurely terminated.

By way of an example, consider the Maclaurin series for e^x

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \tag{2.8}$$

This series has an infinite number of terms but when using it to calculate e^x, only a finite number of terms is actually considered to calculate e^x.

In this example, if only the first three terms are used to calculate e^x, such that:

$$e^x \approx 1 + x + \frac{x^2}{2!} \tag{2.9}$$

TABLE 2.5
Truncation Errors for $e^{1.6}$ Using Maclaurin Series

Number of Terms Considered	$e^{1.6}$	Relative Approximate Error	Absolute Approximation Error
1	1.00000	0.00000	—
2	2.60000	1.60000	61.53846
3	3.88000	1.28000	32.98969
4	4.56267	0.68267	14.96201
5	4.83573	0.27307	5.64685
6	4.92311	0.08738	1.77492
7	4.94642	0.02330	0.47108
8	4.95174	0.00533	0.10756
9	4.95281	0.00107	0.02151
10	4.95300	0.00019	0.00382

A truncation error for this approximation will occur as the following terms will not be considered in the calculation.

The question is raised, how can this truncation error be controlled?

The concept of relative approximate error can be used to see how many terms need to be considered to minimise the error. It will not be solved unless an infinite number of terms are considered; therefore, the engineer has to be realistic and determine the minimum number of terms that have to be considered to give the level of accuracy that can be acceptable.

As an example, calculate $e^{1.6}$ using the Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \quad (2.10)$$

$$e^{1.6} = 1 + 1.6 + \frac{1.6^2}{2!} + \frac{1.6^3}{3!} + \cdots \quad (2.11)$$

Using a hand calculator, the inbuilt function gives $e^{1.6} = 4.9530324244$ to 10 decimal places.

Table 2.5 tabulates the errors resulting from the series after a certain number of elements, and Figure 2.6 depicts the convergence of the series $e^{1.6}$ towards a finite value.

From the table, it will be seen that the error reduces as the number of terms are increased, and the series at element number 7 will have an error less than 1.0%.

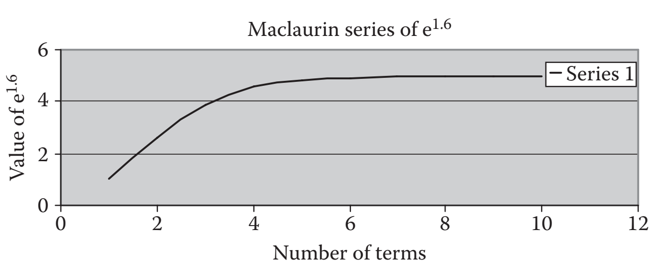


FIGURE 2.6 Showing the convergence of $e^{1.6}$ with an increase in the number of terms used.

2.5 ERRORS ARISING FROM DIFFERENTIATION

The majority of errors that arise from differentiation occur due to the equation being 'ill conditioned'.

As an example of ill conditioning, consider the function

$$f(x) = \frac{1}{(x-1)}$$

when

$$f(1.1), f(x) = 10$$

$$f(1.01), f(x) = 100$$

and

$$f(1.001), f(x) = 1000$$

It is clearly seen that a small change in 'x' of less than 0.1 will result in a rapid change in 'f(x)' approaching 1000. Therefore, evaluating f(x) as 'x' approaches 1.00, f(x) will tend to ∞ making this type of equation very difficult to solve and will introduce errors in its solution.

Consider the equation $f(x) = \sqrt{x}$. When

$$f(1.1), f(x) = 1.04880$$

$$f(1,01), f(x) = 1.00498$$

and

$$f(1.001), f(x) = 1.000499$$

This equation is considered well-conditioned and will yield sensible results with minimum errors.

2.6 INTEGRATION ERRORS

Consider the equation

$$y = x^2 \tag{2.12}$$

It is required to determine the area under the curve between the values x = 3 and x = 9.

Using classical calculus for the solution, consider the integral

$$y = \int_3^9 x^2 \cdot dx \tag{2.13}$$

Using classical integral calculus for the solution:

$$\left[\frac{9^3}{3}\right] - \left[\frac{3^3}{3}\right]$$

Area = 234 units

2.7 SERIES

The study of infinite series is important in the understanding of numerical analysis. Of particular, importance is the 'power series' for two reasons:

1. The evaluation of functions and definite integrals. The subroutines used in computers and pocket calculators are based on series expansions for evaluating functions such as sin x and e^x.
2. Many numerical methods including the solution of differential equations are based on the Taylor series expansion of an arbitrary function.

An infinite series, in general, is an expression in the form

$$u_1 + u_2 + u_3 + u_4 + \cdots + u_n \tag{2.14}$$

where the u_n terms are formed in accordance with a definite rule. The partial sum of the first 'n' terms:

$$s_n = \sum_{i=1}^{n} u_i \tag{2.15}$$

If the limit of s_n as n approaches infinity exists and is finite, the series is said to converge. Conversely, if the limit exists but is infinite, the series diverges. Converging series is the current concern.

A series is said to be absolutely convergent when it remains convergent where all the terms are made positive. Cauchy ratio test provides a simple convergence check. The series will converge if the ratio:

$$R = \lim_{n\to\infty} \left| \frac{u_n + 1}{u_n} \right| \tag{2.16}$$

is less than 1.0 and diverges if the ratio is greater than 1.0. Additional information regarding convergent series will be found in any standard calculus text.

2.8 NEWTON–RAPHSON METHOD

It often becomes necessary to solve algebraic equations of the third or higher orders as well as transcendental equations. To find the roots of these equations often involves a very inefficient trial and error process. The method being described in this section is often known as the Newton–Raphson method, after I. Newton (1666) and J.R. Raphson (1690) who improved on the method.

The Newton–Raphson method is a root-finding process using the first few terms of the Taylor series of a function f(x) in the vicinity of a suspected root and uses an iteration procedure requiring an initial approximation to the solution; the approximation is improved following each iteration. It is a characteristic of iterative methods that minor errors at any step will usually be corrected in the following steps.

2.8.1 Demonstration of the Method

Consider a non-linear equation

$$f(x) = 0 \tag{2.17}$$

The Taylor series representation of f(x) in the vicinity of a point x_0 is

$$f(x) = f(x_0) + (x - x_0)f'(x) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots \tag{2.18}$$

If $x = x_1$ is a solution of Equation 2.17, then $f(x_1) = 0$ and Equation 2.18 becomes:

$$0 = f(x_0) + (x_1 - x_0)f'(x_0) + \frac{1}{2}(x_1 - x_0)^2 f'(x_0) + \cdots \tag{2.19}$$

When the term $(x_1 – x_o)$ becomes sufficiently small, powers of $(x_1 – x_o)$ higher than the first can then be ignored and Equation 2.19 reduces to

$$0 = f(x_0) + (x_1 - x_0)f'(x_0) \tag{2.20}$$

The solution of this equation for x_1 gives the relationship which is then used to find the roots of Equation 2.17:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.21}$$

Using this expression as a recurrent relationship for determining improved values of x_1, each based on a previous approximation x_0, it is possible to evaluate the real roots of Equation 2.17 to any required accuracy.

When Equation 2.21 is used in this way, it is written as

$$x_{n-1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots \tag{2.22}$$

The steps used in the application of Newton's method are summarised as follows:

1. Tabulate or graph f(x) versus x to find an approximate value x_0 of a real root.
2. This starting point can be improved by calculating a corrected value x_1 from Equation 2.21.
3. Successive corrections are made in the same manner using Equation 2.22.
4. The solution is complete when the term $f(x_n)/f'(x_n)$ has no effect on the last decimal place to be retained in x_{n+1}.

EXAMPLE 2.3

The equation $x^3 – 3x – 4 = 0$ is known to have a root at approximately $x = 2$. It is required to find a more accurate root.

SOLUTION

Now, $f(x_0) = x^3 – 3x – 4$; therefore, $f'(x) = 3x^2 – 3$.

A first approximation will be $x_0 = 2$; then

$$f(x_0) = f(2) = -2 \quad \text{and} \quad f'(x_o) = f''(2) = 9$$

A more improved approximation (x_1) will be

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{(-2)}{9}$$

$$= 2.22222$$

Therefore,

$$x_0 = 2.00000; \quad x_1 = 2.22222$$

For the next iteration starting from x_1

$$x_2 = 2.22222 - \frac{0.30724}{11.81479}$$
$$x_2 = 2.19622$$

A further iteration produces

$$x_3 = 2.19622 - \frac{0.00455}{11.47015}$$
$$x_3 = 2.19582$$

and a further iteration results in

$$x_4 = 2.19622 - \frac{0.0000010361}{11.46492}$$
$$x_4 = 2.19582$$

this result is similar to 5 decimal places as x_4.
Hence, the required solution is 2.19582 to 5 decimal places.

It is seen that the method is simple but effective and can be repeated with each repetition or iteration giving a result closer to the required root.

The method lends itself to solution using a spreadsheet.

2.9 ITERATIVE METHODS FOR SOLVING LINEAR EQUATIONS

2.9.1 Gauss Elimination Method

One of the problems with using the Gauss elimination method is that it is sensitive to rounding errors and the method does not offer any refinement to reduce these errors. Also, when the number of simultaneous equations becomes large, that is, 10 or more, the process then becomes laborious.

There are two iterative techniques currently used by engineers:

1. Jacobi iterative method
2. Gauss–Seidel method

2.9.2 Jacobi Iterative Method

Carl Gustav Jacob Jacobi (1804–1851) developed an iterative method for matrices which will be studied in this section.

This method makes two assumptions:

1. That the system of equations given by him has a unique solution

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
&. \\
&. \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n a
\end{aligned}$$

2. And the coefficient matrix 'A' has no zeros on the main diagonal. If any of the diagonal values a_{11}, a_{22}, a_{33},…, a_{nn} are zero, then the rows or columns will need to be interchanged to obtain a coefficient matrix that has nonzero values on the main diagonal.

To review the Jacobi method, consider the Equation 2.23, solve the first equation for x_1, the second equation for x_2 and so on, to x_n as follows:

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \\ x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \\ &: \\ &: \\ x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{nn}x_{n-1}) \end{aligned} \tag{2.24}$$

The next step is to make an initial approximation of the solution,

$$(x_1, x_2, x_3, \ldots, x_n), \tag{2.25}$$

these values of x_1 are substituted into the right-hand side of the rewritten equations to obtain the first approximations. After this procedure has been completed, one iteration is achieved. The second iteration is performed in exactly the same way, but this time substituting the recalculated values for x_1, x_2 x_3, …, x_n into the right-hand side of the equation.

This procedure is then repeated forming a sequence of approximations until they begin to converge to the actual solution. The procedure is illustrated in Example 2.4.

EXAMPLE 2.4

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$12x_1 - 3x_2 + x_3 + 4x_4 = 25$$

$$x_1 + 15x_2 - 7x_3 - x_4 = 6$$

$$4x_1 + x_2 - 20x_3 + 5x_4 = -34$$

$$2x_1 - 8x_2 + x_3 + 10x_4 = 29$$

Solution

To begin, write the above equations in the following form using the rules of algebra;

$$x_1 = \left(\frac{1}{12}\right)(25 + 3x_2 - x_3 + 4x_4)$$

$$x_2 = \left(\frac{1}{15}\right)(6 - x_1 + 7x_3 + x_4)$$

$$x_3 = \left(\frac{1}{20}\right)(34 + 4x_1 + x_2 + 5x_4)$$

$$x_4 = \left(\frac{1}{10}\right)(29 - 2x_1 + 8x_2 - x_3)$$

Because the actual solution is unknown, choose as a convenient initial approximation:

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0$$

The first approximation is

$$x_1 = \left(\frac{1}{12}\right)[25 + 3(0) - (0) + 4(0)] = 2.08333$$

$$x_2 = \left(\frac{1}{15}\right)[6 - (0) + 7(0) + (0)] = 0.400$$

$$x_3 = \left(\frac{1}{20}\right)[34 + 4(0) + (0) + 5(0)] = 1.700$$

$$x_4 = \left(\frac{1}{10}\right)[29 - 2(0) + 8(0) - (0)] = 2.900$$

These values are now substituted into the right-hand side of the equations to get

$$x_1 = \left(\frac{1}{12}\right)[25 + 3(0.400) - 1(700) + 4(2.900)]$$

$$x_2 = \left(\frac{1}{15}\right)[6 - (0.400) + 7(1.700) + (2.900)]$$

$$x_3 = \left(\frac{1}{20}\right)[34 + 4(2.0833) + (0.400) + 5(2.900)]$$

$$x_4 = \left(\frac{1}{10}\right)[29 - 2(2.0833) + 8(0.400) - (1.700)]$$

This procedure is continued until it is clearly seen that the values are converging. Table 2.6 shows the results for this example after 13 iterations, where it is clearly seen that the values have converged after 11 iterations with a very inaccurate start value. If more accurate start values had been chosen, then the convergence would have been quicker.

As the last two columns in Table 2.6 are almost identical, therefore it can be concluded that substituting the values for x_1, x_2, x_3 and x_4 back into the original equations will result in

Line 1 = 24.99885, Line 2 = 5.99914, Line 3 = –34.00203, Line 4 = 28.99875

If the iteration had been terminated at the tenth iteration, the results would have been

Line 1 = 24.98414, Line 2 = 6.01237, Line 3 = –33.99865, Line 4 = 28.9612

It can be seen that increasing the number of iterations results in a more refined answer. A choice has to be made on how many iterations will be needed depending upon the degree of accuracy required.

Although these calculations have been completed by hand to five decimal places, it is possible that the procedure can be automated in Excel or similar programs such as MathCad.

TABLE 2.6
Results of Example 2.4

Start Value	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0.00000	2.08333	1.07500	1.27903	1.19117	1.00068	1.05351	1.01851	0.99870	1.00914	1.00085	1.00002	1.00141	0.99984	1.00010	1.00019	0.99994
0.00000	0.40000	1.24778	1.83933	1.77120	1.92935	1.92114	1.96652	1.99548	1.99667	1.99545	2.00012	1.99928	1.99947	2.00010	1.99985	1.99996
0.00000	1.70000	2.86167	2.63572	2.89704	2.98982	2.94386	2.99585	2.99698	2.99200	3.00090	2.99902	2.99904	3.00029	2.99975	2.99992	3.00005
0.00000	2.90000	2.63333	3.39706	3.85209	3.78902	3.94436	3.97982	3.96993	3.99695	3.99631	3.99610	4.00190	3.99924	3.99958	4.00009	3.99985

2.9.3 Gauss–Seidel Method

Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896) proposed a modification to the Jacobi method that carries their names. This method is considered no more difficult to use than the Jacobi method, and it often requires fewer iterations to obtain the same degree of accuracy.

With the Gauss–Seidel method, when the value for x_1 is determined from the first equation, its value is then used in the second equation to obtain the new x_2. These new values are then used in the third equation obtaining the new x_3; this procedure is then repeated for the remaining equations.

EXAMPLE 2.5

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$12x_1 - 3x_2 + x_3 + 4x_4 = 25 \quad (1)$$

$$x_1 + 15x_2 - 7x_3 - x_4 = 6 \quad (2)$$

$$4x_1 + x_2 - 20x_3 + 5x_4 = -34 \quad (3)$$

$$2x_1 - 8x_2 + x_3 + 10x_4 = 29 \quad (4)$$

As with Example 2.4, the equations are rearranged as follows:

$$x_1 = \left(\frac{1}{12}\right)(25 + 3x_2 - x_3 + 4x_4) \quad (5)$$

$$x_2 = \left(\frac{1}{15}\right)(6 - x_1 + 7x_3 + x_4) \quad (6)$$

$$x_3 = \left(\frac{1}{20}\right)(34 + 4x_1 + x_2 + 5x_4) \quad (7)$$

$$x_4 = \left(\frac{1}{10}\right)(29 - 2x_1 + 8x_2 - x_3) \quad (8)$$

The procedure used for the Jacobi iteration is similar but with a small change.

In this example taking $x_2 = 0$, $x_3 = 0$ and $x_4 = 0$, a value for $x_1 = 2.08333$ is found from substitution in Equation 5.

In Equation 6 to calculate x_2, substituting the value for $x_1 = 2.08333$ together with $x_3 = 0$ and $x_4 = 0$, a value for $x_2 = 0.26111$ is obtained.

To calculate x_3, substituting the revised value for $x_1 = 2.14826$ and $x_2 = 0.26111$ into Equation 7, the value for $x_3 = 2.12972$ is obtained.

To calculate x_4, after substituting the above values for x_1, x_2 and x_3 into Equation 8, the value for $x_4 = 2.46626$ is obtained.

From this point onwards, the iteration procedure is identical to that for the Jacobi method.

The results are shown in Table 2.7, and it will be noted that in this method the values quickly begin to converge on the solution. This does not always happen so quickly unless, as in this example, the coefficients on the main diagonal of the matrix are the largest in each row.

The results also show that using the same number of iterations as the Jacobi method, the results are more accurate.

TABLE 2.7
Results of Example 2.5

Gauss–Seidel Method

Start Value	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2.08333	2.08333	2.14826	1.97113	1.39535	1.11312	1.13571	1.03553	1.01530	1.01876	1.00212	1.00266	1.00236	0.99994	1.00051	1.00025	1.00009
0.00000	0.26111	0.26111	1.25065	1.43294	1.82066	1.89952	1.92759	1.98364	1.98516	1.99146	1.99867	1.99768	1.99914	1.99988	1.99964	1.99998
0.00000	0.00000	2.12972	2.14271	2.77332	2.87372	2.88614	2.98375	2.97945	2.98639	2.99949	2.99643	2.99867	3.00005	2.99940	2.99993	2.99991
0.00000	0.00000	0.00000	2.46626	3.29202	3.48995	3.84653	3.90386	3.93659	3.98591	3.98574	3.99280	3.99876	3.99781	3.99932	3.99986	3.99996

2.10 NON-LINEAR EQUATIONS

Section 2.8 introduced and demonstrated two methods for obtaining the real roots of linear equations. In this section, the procedure is extended to that of finding the roots of non-linear equations using two iterative procedures. The first iteration technique uses a variant of Newton's method and the second is a direct iteration method.

There are a number of physical examples demonstrating non-linearity behaviour, including the motion of a pendulum or the electrical AC waveform, and in control theory, the behaviour of an aerial dish when subjected to wind gusts.

Consider the following two equations:

$$F(x, y) = 0 \tag{2.26a}$$

$$G(x, y) = 0 \tag{2.26b}$$

These two equations define a curve in the x–y plane. Let the problem be to determine the point or points of intersection of these two curves.

Initially as a rough guide to help understand the problem, a graph of the two equations could be drawn to identify the approximate co-ordinates of the points, and once they have been found, the accuracy may be improved using either of the two techniques discussed in this section.

2.10.1 Newton's Method

In Section 2.7, Newton's method was introduced for the solution of one variable; this method can be adopted to a system of equations comprising several variables. As discussed in that section, the method is based on the Taylor series expansion of a function.

A function of two variables, $F(x, y)$, may be expanded within the neighbourhood of a point, (x_0, y_0).

$$\begin{aligned} F(x, y) &= F(x_0 + \alpha \cdot y_0 + \beta) \\ &= F(x_0, y_0) + \alpha F_x (x_0, y_0) + \beta F_y(x_0, y_0) + \cdots \end{aligned} \tag{2.27}$$

where α and β are small increments applied to x and y. The subscripts x and y indicate partial differentiation with respect to x and y. Hence, the notation $F_x(x_0, y_0)$ indicates the partial derivative of $F(x,y)$ with respect to x evaluated at $x = x_0$ and $y = y_0$.

Following the expansion of the original equations in this form, only the linear terms in α and β are retained,

$$\begin{aligned} F(x, y) &= F(x_o, y_o) + \alpha F_x(x_o, y_o) + \beta F_y (x_o, y_o) \\ G(x, y) &= G(x_o, y_o) + \beta G_x(x_o, y_o) + \beta G_y (x_o, y_o) \end{aligned} \tag{2.28}$$

From Equation 2.26a,

$$F(x, y) = 0$$

Therefore, Equations 2.26a, 2.26b and 2.27 become:

$$\alpha F_x + \beta F_y = -F \tag{2.29}$$

$$\alpha G_x + \beta G_y = -G \tag{2.30}$$

where all the functions are evaluated at (x_o, y_o).

If (x_o, y_o) is an initial approximation to the solution, Equation 2.23 can be used to improve the approximation by solving for α and β, then calculating:

$$x_1 = x_o + \alpha$$

$$y_1 = y_o + \beta \tag{2.31}$$

as an improved approximation. This process can be continued until successive approximations only differ by a very small amount.

As in this example only two equations are considered, Cramer's rule can be applied to solve for α and β at each step.

$$\alpha = \frac{\begin{vmatrix} -F & F_y \\ -G & G_y \end{vmatrix}}{D} \quad b = \frac{\begin{vmatrix} F_x & -F \\ G_x & -G \end{vmatrix}}{D} \quad \text{where } D = \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \tag{2.32}$$

When D disappears in the neighbourhood of a solution, there may be multiple solutions, several solutions close together or no solution at all.

EXAMPLE 2.6

Consider the following set of equations to demonstrate the method:

$$y = \sin x$$

$$x^2 + y^2 = 1 \tag{2.33}$$

Solution

For these equations

$$F = y - \sin x \qquad F_x = -\cos x \qquad F_y = 1$$
$$G = x^2 + y^2 - 1 \qquad G_x = 2x \qquad G_y = 2y$$

A plot of these equations indicate that an intersection occurs in the first quadrant close to x = 0.7 and y = 0.7.

$$F = 0.056 \qquad F_x = -0.765 \quad F_y = 1.000$$

$$G = -0.020 \quad G_x = 1.400 \quad G_y = 1.400$$

Substituting in Equation 2.25

$$D = -2.471, \quad \alpha = \frac{-0.0984}{-2.471} = 0.040, \quad \beta = \frac{0.0631}{-2.471} = -0.026$$

From Equation 2.24, the improved approximation will be

$$x_1 = 0.700 + 0.040 = 0.740$$

$$y_1 = 0.700 - 0.026 = 0.674$$

Continuing with this procedure using x_1 and y_1,

$$F = 0.000 \quad F_x = -0.738 \quad F_y = 1.000$$

$$G = 0.002 \quad G_x = 1.480 \quad G_y = 1.348$$

and

$$D = -2.475, \qquad \alpha = \frac{0.002}{-2.475} = -0.001, \qquad \beta = \frac{0.0015}{-2.475} = 0.001$$

hence,

$$x_2 = 0.740 - 0.001 = 0.739$$

$$y_2 = 0.674 - 0.001 = 0.673$$

A further application of the procedure gives

$$x_3 = 0.739$$

$$y_3 = 0.674$$

For these values, F = 0.000 and G = 0.000. Therefore, to three decimal places, the solution to the problem is

$$x = 0.739$$
$$y = 0.674$$

3 Properties of Sections and Figures

The figures in this chapter list the properties for both plane shapes and solid shapes that are most commonly used in engineering design.

The figures are in two parts:

Figure 3.1 considers the properties of plane areas.
Figure 3.2 lists the properties of solid shapes.

The equations shown are for the following:

3.1 CENTROID C_X, C_Y, C_Z

The centroid of an area is the point which is the average distance from all the points on the surface of the figure act.

C_x, C_y and C_z are the centroid co-ordinates.

3.2 MOMENT OF INERTIA/SECOND MOMENT OF AREA

This is a geometric property of the shape, usually associated with the cross section of a beam. It is an indication of the beam's ability to resist bending: the smaller the value of the moment of inertia, the more the beam will deflect; and conversely, the greater the value, the less the beam will deflect less.

The moment of inertia will pass through the centroid of the area and this will be the datum axis for the section. If the axis does not pass through this datum axis, it is possible for the revised moment of inertia to be recalculated for the new axis position.

3.3 POLAR MOMENT OF INERTIA OF A PLANE AREA

The polar moment of inertia will relate to an axis that is perpendicular to the plane of the area.

If all of the area is considered to comprise an infinitely small number of small areas (da), then the polar moment of inertia will be the sum of all these areas multiplied by r^2, where r is the radius of da from the perpendicular axis.

The polar moment of inertia is the sum of any two moments of inertia about an axis acting at 90° to each other in the case of a plane area:

$$J = I_{xx} + I_{yy}$$

where I_{xx} and I_{yy} are the moments of inertia for axes through the centroid in directions x and y.

For a plane solid, it will be the sum of any three moments of inertia, for example, where I_{xx}, I_{yy} and I_{zz} are the moments of inertia for axes through the centroid 'c' in directions x, y and z.

$$J = I_{xx} + I_{yy} + I_{zz}$$

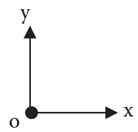
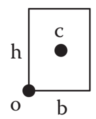
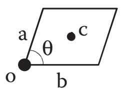
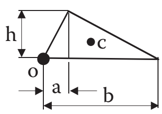
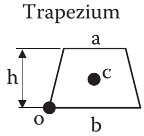
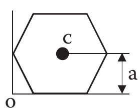
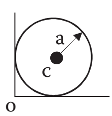
	**Area = A**	**C_x C_y**	**I_{xx}**	**I_{yy}**
Rectangle 	$b \cdot h$	b/h $h/2$	$A \cdot h^2/12$	$A \cdot b^2/12$
Parallelogram 	$a \cdot b \cdot \sin\theta$	$(b + a\cos\theta)/2$ $a \cdot \sin\theta/2$	$A(a \cdot \sin\theta)^2/12$	$A(b^2 + a^2\cos 2\theta)/12$
Triangle 	$b \cdot h/2$	$(a + b)/3$ $h/3$	$A \cdot h^2/18$	$A(b^2 - ab + a^2)/18$
Trapezium 	$h(a + b)/2$	$C_y = \frac{h(2a + b)}{3(a + b)}$	$\frac{A \cdot h\ (a^2 + 4ab + b^2)}{18(a + b)^2}$	—
Hexagon 	$6a^2 \cdot \tan(30°)$ $= 3.464\ a^2$	$a/\cos(30°)$ $= 1.156\ a$	$\frac{A}{12}\left(\frac{a^2[1 + 2\cos^2(30°)]}{\cos^2(30°)}\right)$ $= 0.962\ a^4$	$\frac{A}{12}\left(\frac{a^2[1 + 2\cos^2(30°)]}{\cos^2(30°)}\right)$ $= 0.962\ a^4$
Circle 	πa^2	$C_x = a$ $C_y = a$	$\frac{A \cdot a^2}{4}$	$\frac{A \cdot a^2}{4}$

FIGURE 3.1 Areas, volumes, centroids and moments of inertia for plane figures.

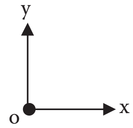
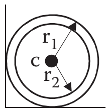
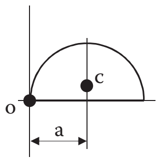
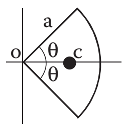
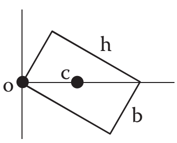
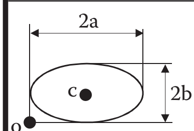
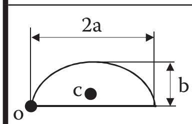
	**Area = A**	**C_x C_y**	**I_{xx}**	**I_{yy}**
Annulus 	$\pi(r_1^2 - r_2^2)$	$C_x = r_1$ $C_y = r_1$	$\frac{\pi(r_1^4 - r_2^4)}{4}$	$\frac{\pi(r_1^4 - r_2^4)}{4}$
Semicircle 	$\frac{\pi a^2}{2}$	$C_x = a$ $C_y = \frac{4a}{3 \cdot \pi}$	$\frac{Aa^2(9\pi^2 - 64)}{36\pi^2}$ $= 0.1098\, a^4$	$\frac{Aa^2}{8}$
Sector of a circle 	$a^2 \cdot \theta$	$C_x = \frac{2 \cdot a \cdot \sin\theta}{3 \cdot \theta}$ $C_y = 0$	$\frac{Aa^2}{4\theta}[\theta - \sin(\theta) \cdot \cos(\theta)]$	$\frac{Aa^2}{4\theta}\left[\theta + \sin(\theta) \cdot \cos(\theta) - \frac{16\sin^2\theta}{9\theta}\right]$
Rectangle 	$b \cdot h$	$C_x = \frac{\sqrt{(b^2 + h^2)}}{2}$ $C_y = 0$	$\frac{Ab^2 \cdot h^2}{6(h^2 + b^2)}$	$\frac{A(h^4 \cdot b^4)}{12(h^2 + b^2)}$
	$\pi \cdot a \cdot b$	$C_x = a$ $C_y = b$	$\frac{A \cdot b^2}{4}$	$\frac{A \cdot a^2}{4}$
	$\frac{A \cdot b}{2}$	$C_x = a$ $C_y = \frac{4 \cdot b}{3 \cdot \pi}$	$\frac{A \cdot b^2(9\pi^2 - 64)}{36\pi^2}$	$\frac{A \cdot a^2}{4}$

FIGURE 3.1 (Continued) Areas, volumes, centroids and moments of inertia for plane figures.

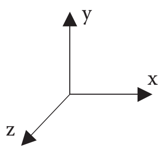
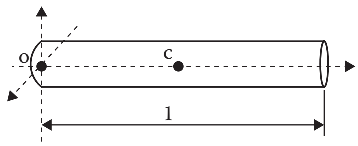
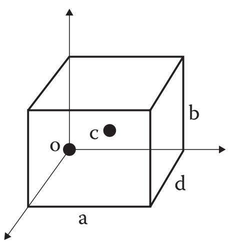
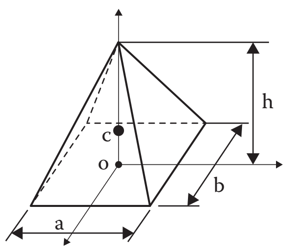
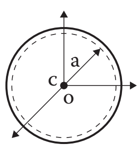
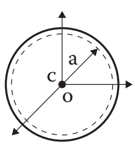
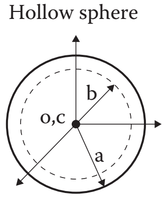
Solid	Area / Volume	C_x / C_y / C_z	I_{xx}	I_{yy}	I_{zz}
Solid 	**Area** **Volume**	C_x C_y C_z	I_{xx}	I_{yy}	I_{zz}
Uniform rod 	$\frac{\pi \cdot a^2}{4}$ $\frac{\pi \cdot a^2}{4} . 1$	1 0 0	0	$\frac{m \cdot 1^2}{12}$	$\frac{m \cdot 1^2}{12}$
Rectangular prism 	$2(a \cdot b + a \cdot c + b \cdot c)$ $a \cdot b \cdot c$	$\frac{a}{2}$ $\frac{b}{2}$ $\frac{c}{2}$	$\frac{m(b^2 + c^2)}{12}$	$\frac{m(c^2 + a^2)}{12}$	$\frac{m(a^2 + b^2)}{12}$
Right rectangular pyramid 	— $\frac{2 \cdot \pi \cdot a \cdot b \cdot c}{3}$ $\frac{a \cdot b \cdot h}{3}$	0 $\frac{h}{4}$ 0	$\frac{m(4b^2 + 3h^2)}{80}$	$\frac{m(a^2 + b^2)}{20}$	$\frac{m(4a^2 + 3h^2)}{80}$
Uniform hoop 	0 0	0 0 0	$\frac{m \cdot a^2}{2}$	$\frac{m \cdot a^2}{2}$	$m \cdot a^2$
Spherical shell 	$4 \cdot \pi \cdot a^2$ 0	0 0 0	$\frac{2 \cdot m \cdot a^2}{3}$		
Hollow sphere 	$4 \cdot \pi \cdot a^2$ $\frac{4 \cdot \pi(a^3 - b^3)}{3}$	0 0 0	$\frac{2 \cdot m(a^5 - b^5)}{5(a^3 - b^3)}$		

FIGURE 3.2 Areas, volumes, centroids and moments of inertia for solid figures.

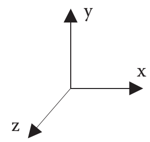
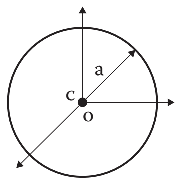
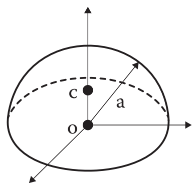
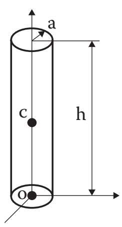
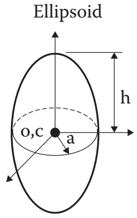
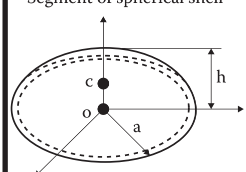
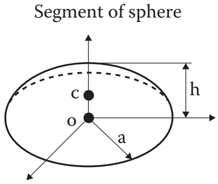
Solid	Area Volume	C_x C_y C_z	I_{xx}	I_{yy}	I_{zz}
Solid 					
Sphere 	$4 \cdot \pi \cdot a^2$ $\frac{4 \cdot \pi \cdot a^3}{3}$	0 0 0		$\frac{2 \cdot m \cdot a^2}{5}$	
Hemisphere 	$2 \cdot \pi \cdot a^2$ $\frac{2 \cdot \pi \cdot a^3}{3}$	0 $\frac{3 \cdot a}{8}$ 0	—	$\frac{2 \cdot m \cdot a^2}{5}$	—
Right circular cylinder 	$2 \cdot \pi \cdot a(h + a)$ $\pi \cdot a^2 \cdot h$	0 $\frac{h}{4}$ 0	$\frac{3 \cdot m(4 \cdot a^2 + h^2)}{80}$	$\frac{3 \cdot m \cdot a^2}{10}$	$\frac{3 \cdot m(4 \cdot a^2 + h^2)}{80}$
Ellipsoid 	— $\frac{2 \cdot \pi \cdot a \cdot b \cdot c}{3}$	0 0 0	$\frac{m \cdot (b^2 + c^2)}{5}$	$\frac{m \cdot (c^2 + a^2)}{5}$	$\frac{m \cdot (a^2 + b^2)}{5}$
Segment of spherical shell 	$2 \cdot \pi \cdot r \cdot h$	0 $\frac{h}{2}$ 0	—	—	—
Segment of sphere 	$2 \cdot \pi \cdot r \cdot h$ $\pi \cdot h^2\left(r - \frac{h}{3}\right)$	0 $\frac{h(4a - h)}{4(3.a - h)}$ 0	$\frac{mh}{80}\left[\frac{A}{(3a - h)^2}\right]$ $A = 240a^3 - 220 \cdot a^2 \cdot h$ $+ 72a \cdot h^2 - 9h^3$	$m\left[a^2 - \frac{3ah}{4} + \frac{3h^2}{20}\right]$ $\times \frac{2h}{3a - h}$	$\frac{mh}{80}\left[\frac{A}{(3a - h)^2}\right]$ $A = 240a^3 - 220 \cdot a^2 \cdot h$ $+ 72a \cdot h^2 - 9h^3$

FIGURE 3.2 **(Continued)** Areas, volumes, centroids and moments of inertia for solid figures.

4 Statics

Statics is the study of rigid bodies that are in equilibrium and concerned with the analysis of loads such as a force or moment/torque acting on a physical body or system, that is, it is in a state where the relative positions of sub-structures do not vary over time or where the structures or sub-structures are moving at a constant velocity without any variation in that velocity.

In statics, it is assumed that the bodies are perfectly rigid without any deformation. In truth, this is not correct. Any physical object will deform or deflect a small amount depending on the level of physical force or load being applied.

These deformations are considered to be insignificant and will not affect the conditions of equilibrium or motion and therefore can be disregarded.

A body can be considered rigid when the relative movement between its constituent parts is negligible.

The basic concepts of statics are

Space: The geometric region occupied by the body where its position can be described by linear and angular measurements relative to a prescribed co-ordinate system.

Time: This is the measure of the succession of events relative to a datum event measured in seconds, minutes or days.

Mass: The measure of the inertia of a body, which is resistant to a change in motion. This is measured in kilogrammes in the SI system of measurement.

Force: The action of an external force acting on a body such as gravitational acceleration or the impact of another body resulting from kinetic energy transfer.

The sections of this chapter will consider:

1. Force, mass and moments
2. Structures and frameworks
3. Vectors and vector analysis

4.1 FORCE, MASS AND MOMENTS

Newton developed the fundamentals of mechanics in which the concepts of space, time and mass are absolute and independent of each other.

Newton's first law: A particle remains at rest or continues to move in a straight line with a constant speed if there is no unbalanced force acting on it (resultant force = 0).

This implies that the net force or net torque on every part of the structure is zero. Therefore, quantities such as stress or pressure can be derived. The net force equalling zero is known as the first condition for equilibrium and the net torque equalling zero is known as the second condition for equilibrium.

For completeness, the two remaining laws are as follows:

Newton's second law: The acceleration of a particle is proportional to the resulting force acting on it and is in the direction of this force.

that is

$$\text{Force} = \text{mass} \times \text{acceleration} \tag{4.1}$$

$$\mathbf{F = ma}$$

Newton's third law: The forces of action and reaction between interacting bodies are equal in magnitude and opposite in direction and act along the same line of action.

4.1.1 System of Units

The units of measurement used in statics are as follows:

Measurement system	Force	Length	Mass	Time
SI Units	Newton (N)	Metre (m)	Kilogram (kg)	Second (s)

$$1\text{N} = (1 \text{ kg})(1 \text{ m/s}^2)$$

Hence, 1 Newton is the force required to give a mass of 1 kg an acceleration of 1 m/s^2.
Weight is a force.
Consider a weight of 1 kg mass:

W = mg (where g is the gravitational constant).
W = (1 kg)(9.81 m/s^2).
W = 9.81 N.

4.1.2 Free-Body Diagrams

A free-body diagram is a very useful method of depicting the interaction of all forces and moments acting on a body at a given situation. It is essentially a sketch of a body that is in equilibrium and entirely separated from its surroundings. Figure 4.1 shows some examples of free-body diagrams.

4.1.3 Forces and Moments

4.1.3.1 Force

A force results from one body acting on another and tends to move it in the direction of its action. A force can be characterised by a localised vector defined by its magnitude, direction and point of application. A single resultant force using the principles of vector addition as shown in Section 4.3.1 can replace a number of forces acting at any point.

Considering Newton's first law (above), it is clear that a body will remain at rest if the resultant force acting on the body is zero. This is referred to as the equilibrium condition and using the Cartesian co-ordinate system can be stated as follows:

$$R = \sum F = 0 \tag{4.2}$$

or

$$\left(\sum F_x\right) \cdot i + \left(\sum F_y\right) \cdot j + \left(\sum F_z\right) \cdot k = 0 \tag{4.3}$$

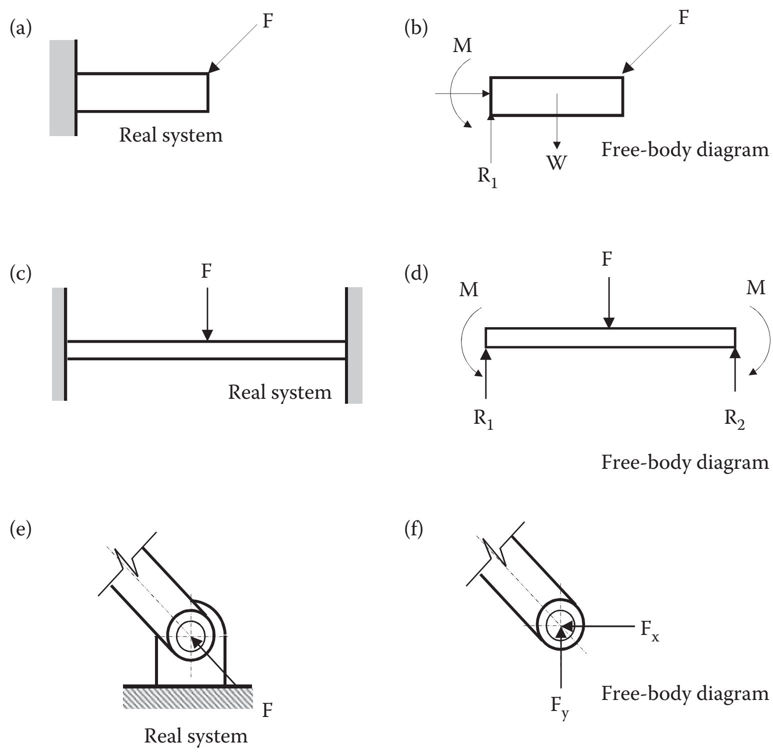


FIGURE 4.1 Examples of free-body diagrams: A cantilever beam (a and b); a beam in encastrè (c and d); frictionless pivot (e and f).

This can be simplified by only considering the scalar identities. …

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0 \tag{4.4}$$

As there are three equations of equilibrium for the three-dimensional case, there can only be three unknowns that can be determined from these equations.

4.1.3.2 Moments

In addition, for the force having the tendency to move a body in the direction of its application, a force can also cause the body to rotate about an axis. The axis of the force may be any line which neither intersects nor is parallel to the line of action of the force. This rotational tendency is referred to as the moment of the force (M) and is the product of the perpendicular distance (d) of the line of action from the point of application of the force (F) multiplied by the magnitude of the force.

The moment of a force is defined in a vector form using determinant algebra (in two- and three-dimensional cases) as

$$M = Fd = \begin{vmatrix} i & j & k \\ d_x & d_y & d_z \\ F_x & F_y & F_z \end{vmatrix} = (d_yF_z - d_zF_y)\cdot i + (d_zF_x - d_xF_z)\cdot j + (d_xF_y - d_yF_x)\cdot k \tag{4.5}$$

$$M_o = d(F_1 + F_2 + F_3 \cdots) = d \cdot F_1 + d \cdot F_2 + D \cdot F_3 \cdots \tag{4.6}$$

Where a number of concurrent forces are applied to a point 'P', the combined moment caused by these forces will be equal to the sum of the moments of these individual forces.

4.1.3.3 Couples

When two forces have an equal magnitude 'F' and are parallel to the line of action but act on either side of the body, the body will rotate about a centre line in either a clockwise or anticlockwise direction. This is defined as a couple and is the magnitude of the individual force multiplied by the distance between the two forces. The direction of the couple is identified by the right-hand rule:

$$\text{Couple} = e \cdot F \tag{4.7}$$

It can also be proved that the moment of the couple is the same magnitude at any location.

4.1.3.4 Rigid-Body Equilibrium

The static equilibrium of a particle is an important concept in the study of statics. A particle is in equilibrium only if all the resultant forces acting upon it are equal to zero and the resultant moments are also equal to zero.

$$\sum F = 0 \quad \sum M = 0 \tag{4.8}$$

Therefore,

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0 \quad \sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0 \tag{4.9}$$

There are six independent equations of equilibrium. The six equations are derived from the free-body diagrams that show all of the applied forces and moments together with all the resulting reaction forces and moments. From these six equations, it is possible to solve for six unknowns. If there are more than six unknowns, then the system is statically indeterminate.

In the special case of a two-dimensional equilibrium, which is often applied to beams and simple structures, the equations for plane static equilibrium are shown below

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_z = 0 \tag{4.10}$$

4.2 STRUCTURES

A structure refers to an assembly of materials whose function is to support a load or loads, be they concrete, brick, metal linkages and so on. The term structure can be applied to an aircraft wing, bridge, building or any form where a load or force is being resisted. The component parts of a loaded structure are in a state of stress that is tensile, compressive or shear. The distribution of these loads or forces is analysed to determine which materials are most appropriate to safely support the load or forces.

Structures are classified into two basic groups:

1. Framed structures
2. Mass structures

The former is manufactured from either separate linkages or plates welded or riveted together to form a lattice. The latter will depend upon the mass of material to provide a resistance to the load, such as a masonry dam to withstand the water pressure in a reservoir.

For the purposes of this discussion, only the former (framed structures) will be considered as the latter is more in the civil engineering domain.

In the notes that follow, the framework or trusses are considered to be manufactured from linkages connected to each other at pin joints that do not transmit moment forces. The connecting linkages can only transmit either tensile forced (ties) or compressive forces (struts). For the purposes of this discussion, it is assumed that the ties and struts do not experience any axial and transverse deflections. Figure 4.2 shows examples of pin-jointed truss structures.

Structures can be classified as either 2-dimensional (plane frames) or 3-dimensional (space frames).

For a 2-dimensional frame, the number of linkages (N) required with J joints will be

$$N = 2 \cdot J - 3 \tag{4.11}$$

And for a 3-dimensional frame, the number of linkages (N) required with J joints will be

$$N = 3 \cdot J - 6 \tag{4.12}$$

Note: It will be clear to the reader that in practice most frame joints are not pinned. The principles described here can be used with reasonable accuracy for a majority of frames where the ratio of the length of the linkage to its depth (L/d) is in excess of 10. In the case of frames that have relatively rigid joints and short linkages, the analysis is statically indeterminate and therefore outside the scope of this discussion.

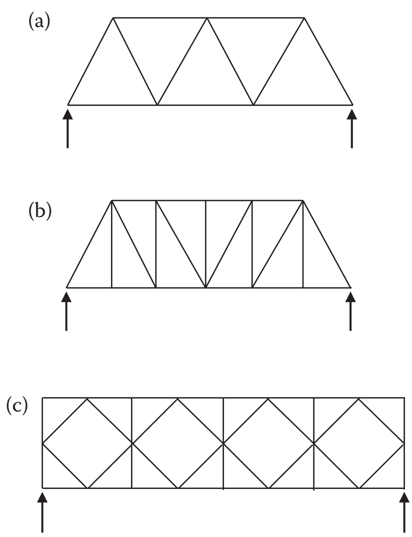


FIGURE 4.2 Types of truss structures. Warren truss (a); Pratt truss (b); Bailey truss (c).

4.2.1 Pin Joint

A pin joint allows the joined members to swivel as opposed to a rigid joint which does not allow any rotation. A rigid joint may be welded, whereas a pinned joint may be a bolt, rivet or any other form of swivel pin.

Figure 4.3 shows two methods of providing a pin joint.

Two important points about a pin joint are

1. The connected members are free to rotate.
2. The force in the member can only carry axial forces or loads.

4.2.1.1 Struts and Ties

Consider a member depicted in Figure 4.4a and b with a pin joint at each end. A pin joint cannot transmit rotation from one member to another as each can only push or pull on the joint along the direction of its length. A member subject to a tension load is known as a 'TIE' and is shown with arrows pointing inwards at each end as in Figure 4.4a. A member in compression is called a 'STRUT' and is shown with arrows pointing outwards at each end as shown in Figure 4.4b.

4.2.1.2 Bow's Notation

When several members of a structure are pinned together and the joints are in equilibrium, the resultant force must be zero. When all the forces are added up as vectors, they must form a closed polygon. If one or more of these are unknown, then it must be the vector that closes the polygon.

Consider three members joined by a pin as shown in Figure 4.5. Only one of these forces is known. A method known as 'Bow's notation' is used to help identify and label each member enabling a polygon (or in this case) a triangle of forces to be drawn. The process is as follows:

1. Label the spaces between each member as shown in Figure 4.5. This diagram is also known as a 'space diagram'.
2. Starting at any space, say 'A', identify each member individually by moving clockwise around the joint, so that the first member is identified as a–b, the next member as b–c and the last as c–a (in this case only).

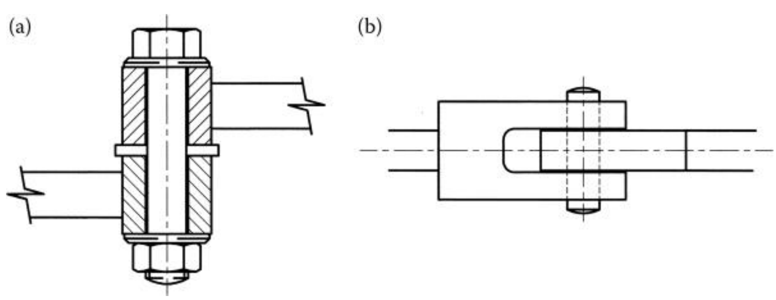


FIGURE 4.3 (a) and (b): Types of pin joints.

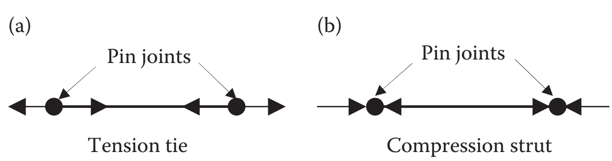


FIGURE 4.4 Ties (a) and struts (b).

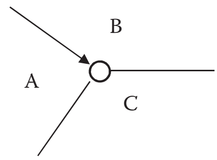


FIGURE 4.5 Bow's notation.

3. Draw the known vector a–b in the same direction as the space diagram. The next vector b–c starts at 'b', but at the moment the length is not known; therefore, draw a line representing b–c. When all the vectors are added together, they must form a closed triangle; hence, c–a will end at 'a'. Complete the line c–a and where the two 'c' lines cross, this will be the point 'c' as shown in Figure 4.6.
4. Finally, transferring the arrows back to the space diagram in the same direction as shown on the triangle of forces (Figure 4.7) where they push onto the pin joint, the member will be in compression and therefore will be a strut. If the arrow pulls away from the pin joint, the member will be in tension and is therefore a tie.

EXAMPLE 4.1

A column (strut) is held vertically by two guy ropes as shown in Figure 4.8a. The maximum allowable compressive force in the strut is 25 kN. Calculate the forces in each rope bearing in mind that the ropes can only be in tension.

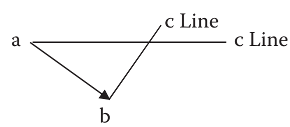


FIGURE 4.6 Vector diagram.

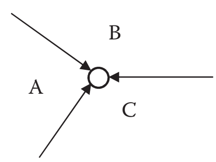


FIGURE 4.7 Triangle of forces.

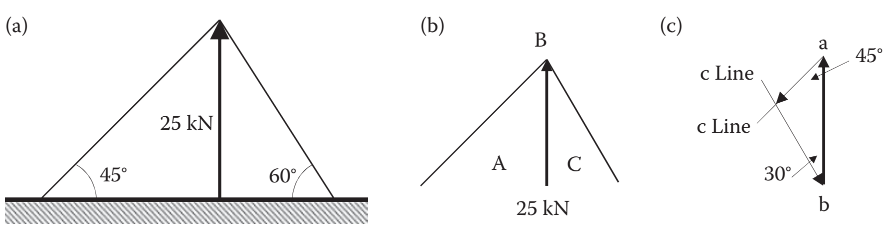


FIGURE 4.8 (a) Example 4.1, (b) Bow's notation, and (c) vector diagram for Example 4.1.

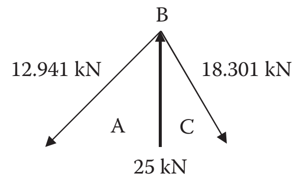


FIGURE 4.9 Summary of forces for Example 4.1.

Solution

Following the procedure as outlined in Section 4.2.1.2, first draw the space diagram (Figure 4.8b) and then complete the triangle of forces (Figure 4.8c).

The forces in each of the ropes will be b–c and c–a. These forces can be found by either drawing to scale or by calculation.

First, ensure that the internal angles of the triangle of forces add up to 180°. In this instance, the angle between vectors ac and bc will be

$$180° - (30° + 45°) = 105°$$

$$\frac{bc}{\sin 45°} = \frac{25}{\sin 105°}$$

$$bc = 18.30 \text{ kN (Tension)}$$

and

$$\frac{ca}{\sin 30°} = \frac{25}{\sin 105°}$$

$$ca = 12.94 \text{ kN (Tension)}.$$

Figure 4.9 summarises these forces.

4.2.2 Solving Forces in Pin-Jointed Frames

There are many examples of lattice work frames that are made up of a number of pin-jointed members such as

- Roof trusses
- Crane jibs
- Bridges
- Electrical pylons

Many of these structures use riveted or bolted joints and are not entirely free to rotate at the joint, but the theory for pin-jointed frames appears to work quite well. Bow's notation is applied to each joint in turn and solving the forces in each member. Transferring the force directions back from the polygon to the framework diagram, it can then be deduced which members are struts or ties. The force direction at the other end of the member can then be determined, and this is what is needed to help solve the forces in the other pin joints.

EXAMPLE 4.2

A pin jointed framework has a load of 250 kN applied to the joint as shown in Figure 4.10a.
Solve the forces and reactions for the framework shown.

Solution

1. Draw the space diagram and then label the spaces as shown in Figure 4.10a using Bow's notation.
2. Solve the known joint using either drawing or trigonometry (Figure 4.10b.)

$$ac = 250\ \text{kN} \times \sin 30° = 125.0\ \text{kN (strut)}$$

$$bc = 250\ \text{kN} \times \cos 30° = 216.5\ \text{kN (strut)}$$

3. Solve for the other joint (Figure 4.10c)

$$bd = 216.5 \times \cos 30° = 187.5\ \text{kN } (R_2)$$

$$cd = 216.5 \times \sin 30° = 108.25\ \text{kN}$$

4. $$ad = 125\ \text{kN} \times \sin 30° = 62.50\ \text{kN } (R_1)$$

$$cd = 125\ \text{kN} \times \cos 30° = 108.25\ \text{kN}$$

The reaction R_2 is easily deduced as the total upward force is 250 kN. Hence, R_2 = (250 kN – 187.5 kN) = 62.50 kN. The solution for the remaining joint is not needed except for completeness: (see Figure 4.10d).

When the reader becomes proficient with this work, it will be found to be more convenient to complete this work in one diagram (see Figure 4.10e).

There are three methods for analysing frameworks:

1. Method of joints
2. Graphical methods
3. Method of sections

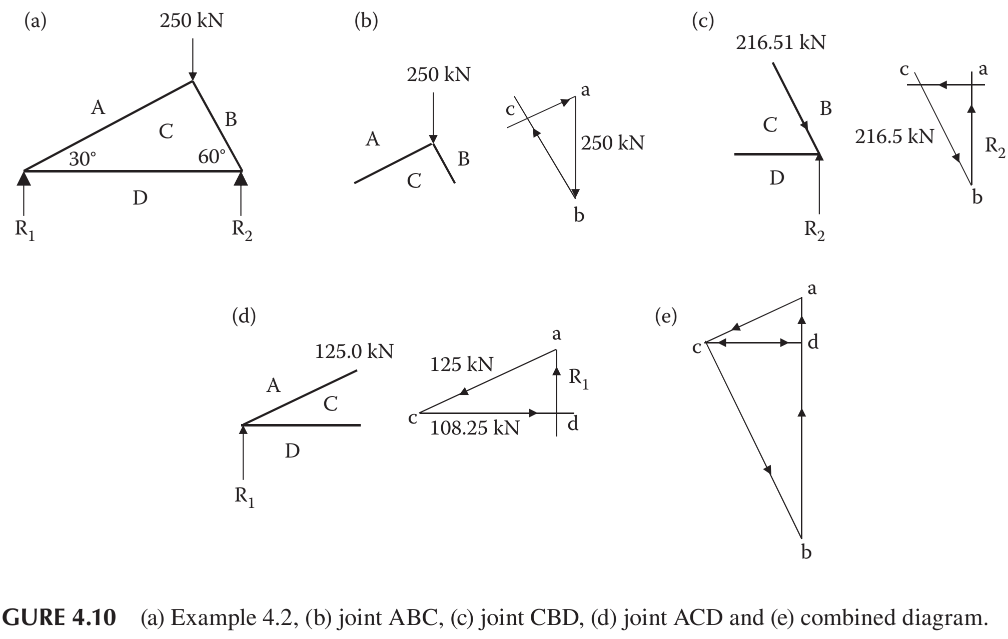


FIGURE 4.10 (a) Example 4.2, (b) joint ABC, (c) joint CBD, (d) joint ACD and (e) combined diagram.

4.2.3 Method of Joints

The method of joints is a technique for finding the internal forces within a framework and it works on the assumption that the framework members are pin connected, making them two force members. Equations of static equilibrium can then be written for each pin joint and the set of equations can be solved simultaneously to determine the forces acting in the framework members.

The biggest problem with the 'method of joints' is the amount of work that has to go into calculating each member force. It may be difficult to slove just for these forces alone; instead, the solution may require more forces than required.

In the real world, the framework members will normally be connected using welded gusset plates; the idealised joint is considered as that connected by a frictionless pin.

Although there will be an error, it is normally considered acceptable when the framework members are long and slender, that is, $L/d > 10$.

The steps in applying the method of joints as applied to the analysis of a plane framework are as follows:

1. Label all the pin joints (A, B, C, … etc.).
2. Draw and label a free-body diagram for the complete framework.
3. Determine the reaction forces using the three equations for static equilibrium applied to the whole framework.
4. Draw a free-body diagram for each pin assembly. This may be one free-body diagram that shows all the pin joint details.
5. If possible, begin solving the equilibrium equations at a joint where only two unknown reactions exist. Work from joint to joint using the criterion of two unknown reactions.
6. Apply the two equations of static equilibrium relating to F_x and F_y, (moments are zero) at each joint to identify the forces in the attached members.

EXAMPLE 4.3

A truss as shown in Figure 4.11 has a force acting at the apex of the truss. Draw the free-body diagram and determine the forces acting within the individual links.

Solution

See Table 4.1 for the link calculations.

4.2.4 Graphical Methods as Applied to a 2-Dimensional Framework

The graphical method for the analysis of frameworks is a convenient method for simple frames and trusses, but as the framework becomes more complex, the method then becomes more complicated and cannot be used for 3-dimensional space frames.

Consider the frame in Figure 4.12.

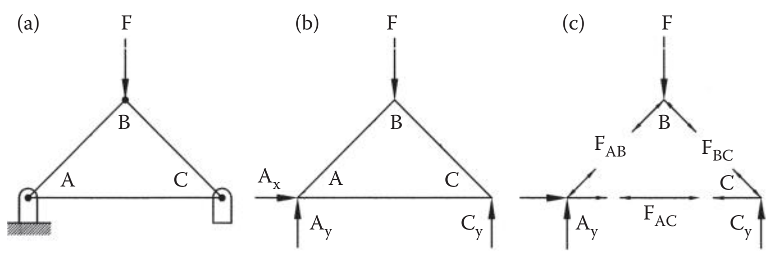


FIGURE 4.11 (a), (b), and (c): Example 4.3.

TABLE 4.1
Calculated Values for Figure 4.12

Figure 4.13a	Figure 4.13b	Figure 4.13c
Angle A & C = 45°	$\Sigma M_z = 0$	*Joint A*
AC = 1000 mm	*Moments about A*	$F_{ab} = 707/1000 = A_y$
AB = 1000_x sin 45°	C_{yx} 1000 = 1000 N × 500	$F_{AB} = A_{yx}$ 1000/707
= 707 mm	C_y = 500 N	= 707 N
F = 1000 N		$F_{AC} = F_{AB} \times 707/1000$
	$\Sigma F_x = 0$	= 707 N
	$A_x = 0$	
		Joint B
	$\Sigma F_y = 0$	$F_{BC\,x}$ 707/1000
	A_y = 1000 N – C_y	= 1000 N – F_{AB}
	= 500 N	707/1000 N
		F_{BC} = 707 N
		Joint C
		F_{AC} = 707 N
		F_{BC} = 707 N

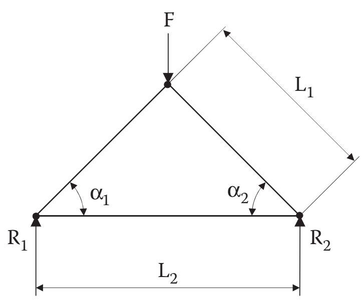


FIGURE 4.12 Basic truss frame.

As with the method of joints procedure, the initial steps is to

- Label all the pin joints.
- Draw and label a free-body diagram for the whole framework.
- Establish the reaction forces.
- Draw a free-body diagram for each individual joint; this may be combined onto a single free-body diagram.
- Produce a force vector polygon for each joint starting with one having no more than two unknowns.
- The force diagrams can be combined into one single diagram [this is also known as a Maxwell diagram (see Figure 4.13)].

4.2.5 Method of Sections as Applied to a Plane Framework

The graphical method demonstrated in Section 4.2.4 can be expanded to cover more complicated figures such as that shown in Figure 4.14. It also can, in some cases, be used to determine the forces in selected bars more rapidly than using the method of joints.

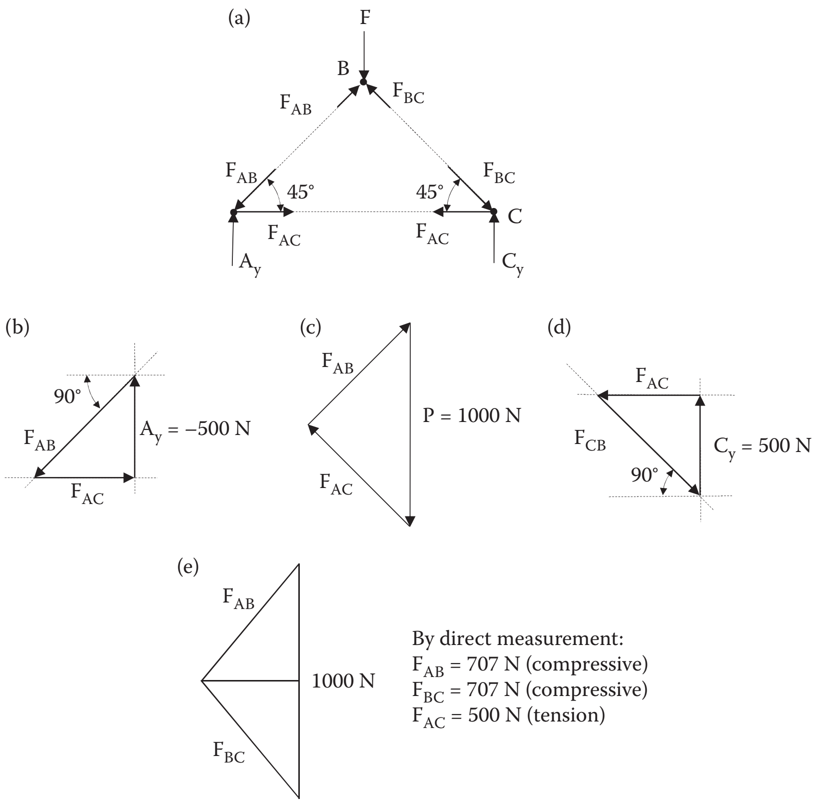


FIGURE 4.13 (a)–(e): Graphical method of joint analysis.

The steps used in this process are similar to that used in Section 4.2.4:

1. Label all pin joints—A, B, C, etc.
2. Draw and label a free-body diagram covering the whole framework.
3. Determine the reaction forces using the three equations for static equilibrium applied to the whole framework.
4. Draw a free-body diagram for a selected part of the frame, which can include two joints breaking the bars under consideration.
5. Calculate the forces in the bars to one side of the 'cut', again using the three equations for static equilibrium.

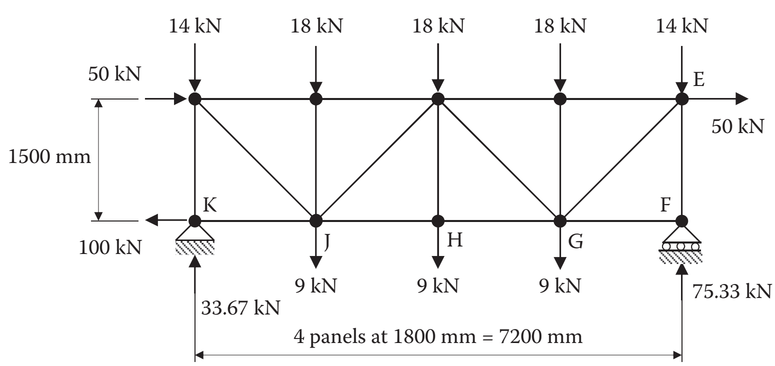


FIGURE 4.14 Warren girder.

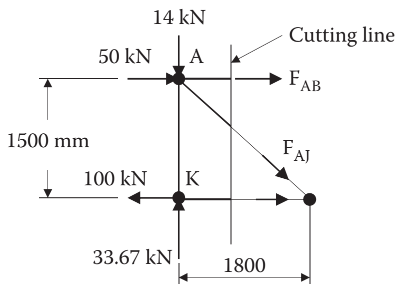


FIGURE 4.15 Free-body diagram for Figure 4.14.

Figure 4.14 shows an example of a Warren-type girder carrying a variety of loads. It is required for the purposes of this exercise to determine the forces in members AB and JK.

Start at a point on the frame where the forces are known (see Figure 4.15).

A cutting line is drawn through the members under consideration and a free-body diagram is constructed for the part of the framework to the left of the cutting line (see Figure 4.15).

The unknown forces are shown in tension.

Taking moments about joint A will eliminate forces F_{AJ} and F_{AB} leaving force $F_{KJ.}$

Sum the moments about A to zero:

$$0 = 100(1.5) – F_{KJ}(1.5) \tag{4.13}$$

therefore, $F_{KJ} = 100$ kN (tension).

Now taking moments about joint J which is outside the free-body diagram, this will eliminate F_{AJ} and F_{KJ} leaving only F_{AB}.

Summing moments about J to zero:

$$0 = 50(1.5) – 14(1.8) + 33.67(1.8) + F_{AB}(1.5)$$

$$= –73.60 \text{ kN (tension)}$$

Summing moments vertically to zero, this will eliminate both F_{KJ} and force F_{AB} leaving F_{AJ}.

$$0 = 33.67 – 14 – F_{AJ}\ (5/7.81)$$

$$F_{AJ} = +30.72 \text{ kN (tension)}$$

$$F_{KA} = –33.67 \text{ kN (compression)}$$

Figure 4.16 shows a typical truss construction.

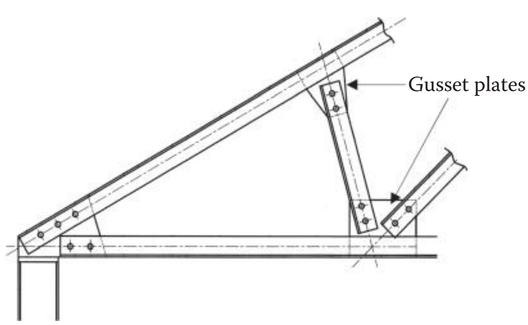


FIGURE 4.16 Typical truss construction.

4.3 VECTORS AND VECTOR ANALYSIS

Vector algebra is particularly important when dealing with problems involving forces, displacements, velocities, accelerations, moments and so on in either two or three dimensions. These quantities have magnitude and direction that define them.

Quantities such as mass, volume, power and so on, however, have magnitude, direction is not involved and these are referred to as scalars.

Vectors do not obey the normal rules of addition and subtraction; they are added or subtracted using the Parallelogram rule.

4.3.1 Vector Addition

Consider two vectors $\vec{R}$ and $\vec{Q}$ acting on a single point: From Figure 4.17a–c, it can be seen that these can be replaced by a single vector $\vec{R}$. By drawing lines parallel to the vectors to complete the parallelogram, the resultant 'R' is found.

In Figure 4.17c, the vector Q has been moved parallel to itself and its tail has been placed to the head of P, and the resultant 'R' is as shown. This is known as 'triangular construction'.

The triangular rule may be applied when it is more convenient to use the parallelogram rule.

Analytically, the magnitude R of the vector R is given by

$$R = \sqrt{P^2 + Q^2 - 2PQ\cos(180 - q)} \quad \text{(cosine rule)} \tag{4.14}$$

4.3.2 Vector Subtraction

Vectors may be subtracted, that is $\vec{P} - \vec{Q}$, by applying the rule thus: Draw $\vec{P}$ and $-\vec{Q}$ and add either the parallelogram or triangular rules to obtain $\vec{R}$ as shown in Figure 4.18b.

4.3.3 Resolving a Vector into Components

Consider a vector $\vec{P}$ as in Figure 4.19 and determine its components.

In this example, there is a requirement to convert the vector P into its components. P_x and P_y are the rectangular components along the Cartesian axes o_x and o_y, and their magnitudes will be

$$P_x = P\cos\theta \quad \text{and} \quad P_y = P\sin\theta \tag{4.15}$$

It also follows that:

$$P = \sqrt{P_x^2 + P_y^2} \quad \text{in magnitude} \tag{4.16}$$

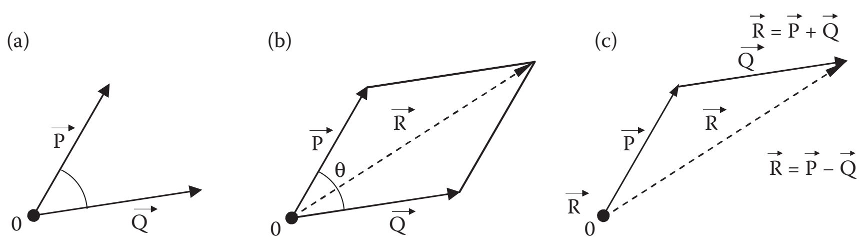


FIGURE 4.17 (a)–(c): Resultant of two forces.

$P \cdot Q = PQ \cos \phi$

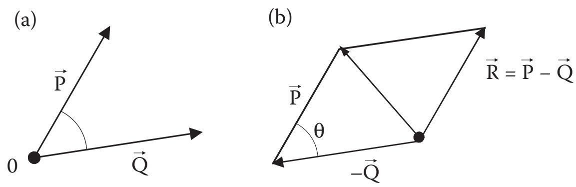


FIGURE 4.18 (a) and (b): Subtracting vectors.

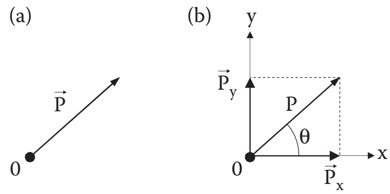


FIGURE 4.19 (a) and (b): Resolving co-ordinates.

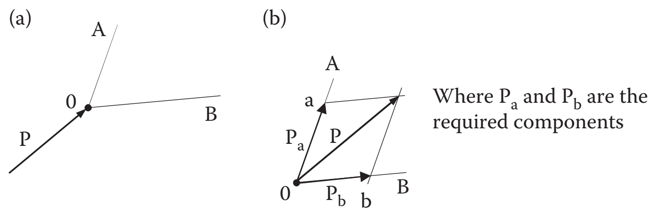


FIGURE 4.20 Space diagram for given force and direction.

Next, consider the case where the co-ordinates are not rectangular:

Figure 4.20 shows a space diagram of a force vector and it is required to determine the respective forces given the directions.

4.3.4 Analytical Determination of the Components of the Vector

Figure 4.21 shows a space diagram giving the force and component directions for the following example:

Let

$$\gamma = 180 - (\alpha + \beta) \tag{4.17}$$

From the sine rule:

$$\frac{P}{\sin \gamma} = \frac{P_a}{\sin \beta} = \frac{P_b}{\sin \alpha} \tag{4.18}$$

From which:

$$P_a = \frac{\sin \beta}{\sin(\alpha + \beta)} P \tag{4.19}$$

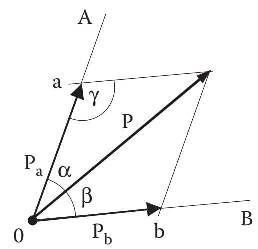


FIGURE 4.21 Space diagram for given force and component directions.

$$P_b = \frac{\sin\alpha}{\sin(\alpha+\beta)} P \tag{4.20}$$

If the co-ordinates are rectangular as in Figure 4.19, then

$$P_a = P \sin\beta \text{ (vertical)} \quad \text{and} \quad P_b = P\cos\beta \text{ (horizontal)} \tag{4.21}$$

Note: In the majority of cases the directions 0A and 0B are perpendicular with each other in which case the angles α and β will be required.

4.3.5 Resultant of a Number of Coplanar Vectors (More than Two Vectors)

Note: For vectors to be concurrent a their lines of action must meet at a point.

The procedure for constructing a vector polygon is as follows (Figure 4.22):

1. Establish an origin '0' and from this draw P_1 parallel to the given vector 'P_1' and drawing the magnitude to scale.
2. From the head of 'P_1', place the tail of 'P_2' and draw the vector 'P_2' parallel to the given vector 'P_2' and in this instance applying a triangular construction.
3. Repeat this procedure for the remaining vectors.
4. The *resultant* (R) will be given by the vector ad.

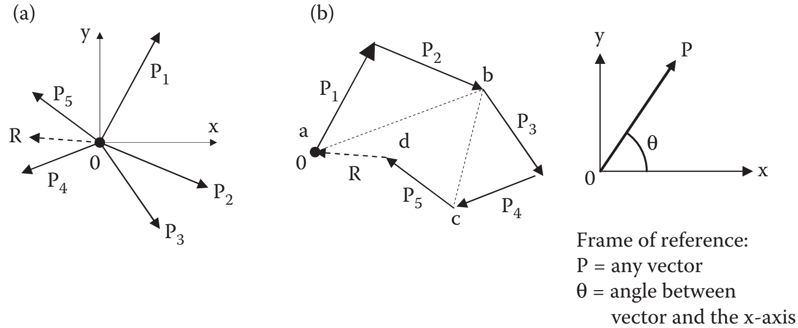


FIGURE 4.22 Construction of vector polygon. Given vectors (a); vector polygon (b).

4.3.6 Analytical Solution to Figure 4.22

Resultant 'R'

1. Magnitude of vector:

$$R = \sqrt{(\sum P \sin\theta)^2 + (\sum P \cos\theta)^2} \tag{4.22}$$

Where

$$\sum P \sin\theta = P_1 \sin\theta_1 + P_2 \sin\theta_2 + \cdots + P_5 \sin\theta_5$$

$$\sum P \cos\theta = P_1 \cos\theta_1 + P_2 \cos\theta_2 + \cdots + P_5 \cos\theta_5$$

2. Direction relative to the 'x'-axis
Let

$$R_x = \sum P \cos\alpha \quad \text{and} \quad R_y = \sum P \sin\alpha$$

$$\alpha = \arctan\left(\frac{R_y}{R_x}\right) = \arctan\left(\frac{\Sigma P \sin\alpha}{\Sigma P \cos\alpha}\right) \tag{4.23}$$

4.3.7 Product of Vectors

4.3.7.1 Multiplication of a Vector 'P' by a Scalar 'K'

The product of a vector with a magnitude of 'P' and direction 'θ' and a scalar is a vector increased in magnitude. Thus, a vector P multiplied by a scalar K will give a vector with a magnitude of KP and the direction remains unaltered at 'θ'. If K is –ve, the sense of the vector is in the reversed direction.

4.3.7.2 Scalar Product of Two Vectors

1. The scalar product of two vectors, as shown in Figure 4.23, is defined as the product of the two magnitudes times the cosine of the angle between the two vectors. The result is a scalar quantity. The scalar product is written with a dot between the symbols, thus:
2. Scalar products are commutative:

$$A \cdot B = B \cdot A$$

3. Scalar products are distributive:

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

4. If $A \cdot B = 0$, then $A = 0$ or $B = 0$ or A and B are at right angles.

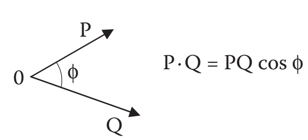


FIGURE 4.23 Product of two vectors.

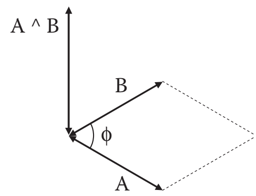


FIGURE 4.24 Vector product.

4.3.8 Vector (or Cross) Product

1. The vector product of two vectors (Figure 4.24) is defined as a vector whose magnitude is equal to the product of magnitudes times the sine of the angle between the vectors. The direction is perpendicular to the plane containing the two vectors and forms a right-handed system.
   The symbol for a vector product is $\wedge$.
   Thus, $A \wedge B = (AB \sin \phi)n$, where $AB \sin \phi$ is the magnitude and n is the unit normal vector to the plane containing A and B (see Figure 4.24).
2. Vector products are not commutative, that is, $A \wedge B \neq B \wedge A$.
   However,

$$A \wedge B = -B \wedge A$$

3. Vector products are distributive provided the order of the vectors is maintained.

$$A \wedge (B + C) = A \wedge B + A \wedge C$$

4. If $A \wedge B = 0$, then $A = 0$ or $B = 0$ or A and B are parallel.

5 Dynamics

5.1 KINEMATICS

Kinematic analysis is the study of the motion of a particle or body without regard to the forces that may be generating the motion. The motion may be either rectilinear or rotary and in mechanical engineering this includes the motion of linkages in mechanisms including that of robotic arms and so on. The definition of a mechanism in this context is rigid bodies connected by joints and is applied to the combination of geometric bodies that constitute a machine or part of a machine. They move with definite relative motions with respect to each other and are often referred to as the geometry of motion.

5.2 NOMENCLATURE

s	linear displacement (m)
v	average velocity (m/s)
a	acceleration (m/s^2)
t	time (s)
u	initial velocity (m/s)
v	final velocity (m/s)
θ	rotation angle (radians)
ω_1	initial angular velocity (rad/s)
ω_2	final angular velocity (rad/s)
η	angular speed (rev/min)
α	angular acceleration (rad/s^2)

5.3 NEWTON'S LAWS OF MOTION (CONSTANT ACCELERATION)

In Chapter 4 (Statics), the reader was introduced to Newton's three laws of motion. In this chapter, the second law is considered as a set of equations. There are four equations and these describe linear and angular motions.

5.3.1 Linear Motion Equations

$$s = ut + \frac{at^2}{2} \tag{5.1}$$

$$v = u + at \tag{5.2}$$

$$v^2 = u^2 + 2as \tag{5.3}$$

$$s = \left(\frac{u + v}{2}\right)t \tag{5.4}$$

$$v = \frac{ds}{dt} \tag{5.5}$$

$$a = \frac{dv}{dt} = v\frac{dv}{ds} \tag{5.6}$$

5.3.2 Angular Motion Equations

$$\theta = \omega_i t + \frac{\alpha t^2}{2} \tag{5.7}$$

$$\omega_f = \omega_i + \alpha t \tag{5.8}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\theta \tag{5.9}$$

$$\theta = \left(\frac{\omega_i + \omega_f}{2}\right)t \tag{5.10}$$

$$\omega = \frac{d\theta}{dt} \tag{5.11}$$

$$\alpha = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} \tag{5.12}$$

These laws will be expanded in the following sections.

5.4 RECTILINEAR MOTIONS

Rectilinear motion (also known as linear motion) is motion in one direction only, that is, along a straight line and can be described mathematically as having one spatial dimension. The motion can be of two types:

- Uniform linear motion with a constant velocity (or zero acceleration)
- Non-linear motion with a variable velocity (or non-zero acceleration)

The motion of a particle along a straight line can be described by its position 'x', which varies with time 't'. A good example of linear motion is a machine tool slide on a lathe.

5.4.1 Uniform Linear Motion

Of all the motions, linear motion is the most basic and according to Newton's first law of motion, providing an object does not experience any net force and will continue to carry on moving in a straight line at a constant velocity until it is subjected to an external force. External forces such as gravity and friction can cause an object to change the direction of its motion such that the motion therefore cannot be described as linear.

In Figure 5.1, the object travels a distance 's' over a time 't' at a uniform velocity.

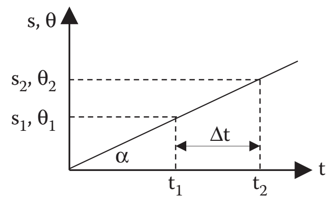


FIGURE 5.1 Linear displacement–time.

5.4.2 Non-Uniform Linear Motion

In linear motion where the object has a non-uniform velocity, the object's velocity is continuously changing, in which case the acceleration will also be changing in a non-linear relationship. An example is if a body is falling to earth, it will fall with an increasing acceleration, and hence the object's velocity will be varying in a non-linear manner.

Figure 5.2 shows the displacement/time curve of such a situation where the displacement 's' is changing in accordance with a specific law and the distance travelled at a particular point can be determined by drawing a tangent to the curve at a time $t = t_1$ and measuring the distance. Knowing the relationship between the distance 's' travelled and the time 't', differentiating 's' with respect to 't' will result with the velocity of the object at that point.

5.4.3 Variable Velocity

Figure 5.3 shows a curve where the linear or angular velocity is changing with respect to time. Differentiating with respect to time will give the linear or angular acceleration.

5.4.4 Variable Acceleration

Consider the case where the acceleration of a body is increasing in a non-linear relationship:

Figure 5.4 depicts the relationship between acceleration 'a' and 'a' with respect to time.

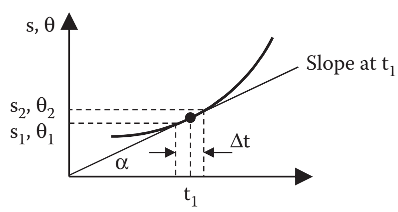


FIGURE 5.2 Variable displacement–time.

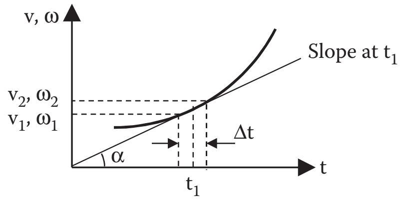


FIGURE 5.3 Velocity–time.

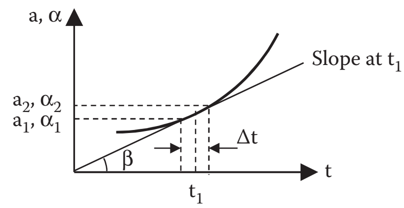


FIGURE 5.4 Acceleration–time.

Linear velocity:

$$v = \frac{ds}{dt} = \dot{s} = \tan \alpha \tag{5.13}$$

Angular velocity:

$$\omega = \frac{d\theta}{dt} = \dot{\theta} = \tan \alpha \tag{5.14}$$

Linear acceleration:

$$a_l = \frac{dv}{dt} = \dot{v} = \tan \beta \tag{5.15}$$

Angular acceleration:

$$a_r = \frac{d\omega}{dt} = \dot{\omega} = \tan \beta \tag{5.16}$$

5.5 CIRCULAR MOTION

5.5.1 Motion on a Circular Path

Figure 5.5 describes the motion of a particle in a circular path with 'ω' given:

$$v = r\omega \tag{5.17}$$

$$a = r\omega^2 = \frac{v^2}{r} \tag{5.18}$$

The acceleration 'a' is towards the centre 'C' and it is referred to as the centripetal acceleration.

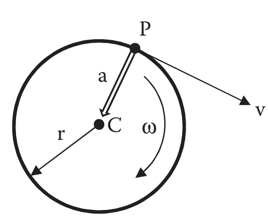


FIGURE 5.5 Motion in a circular path.

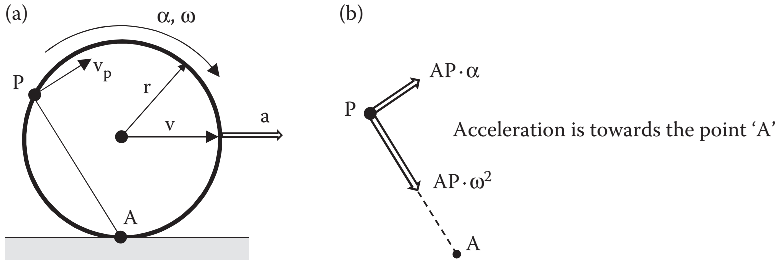


FIGURE 5.6 (a) A particle on a rolling wheel and (b) the vector representation of that point.

5.5.2 Rolling Wheel

Figure 5.6a and b shows a rolling wheel.

With 'v' and 'a' given:

$$\omega = \frac{v}{r} \tag{5.19}$$

$$\alpha = \frac{a}{r} \tag{5.20}$$

At any point 'P' on the wheel rim:

$$v_p = \omega AP \tag{5.21}$$

At the top of the wheel:

$$\text{Velocity} = 2 \cdot v \tag{5.22}$$

5.6 ABSOLUTE AND RELATIVE MOTION

An example of absolute motion can be explained by considering a frame of reference based on the earth, an individual sitting in either a motor vehicle or train carriage; the motion of the individual relative to the earth can then be considered absolute against other individuals or objects within the same frame of reference.

If the reference frame is now changed and is based upon the train carriage, when the individual in the train carriage decides to move along the carriage, the individual's motion will be considered relative against the local reference frame.

5.7 ROTATING UNIT VECTOR

Consider a rotating unit vector e….

See Figure 5.7a:

$$e = i\cos\theta + j\sin\theta \tag{5.23}$$

$$\frac{di}{dt} = \frac{dj}{dt} = 0 \text{ (i and j are unit vectors in x and y directions)} \tag{5.24}$$

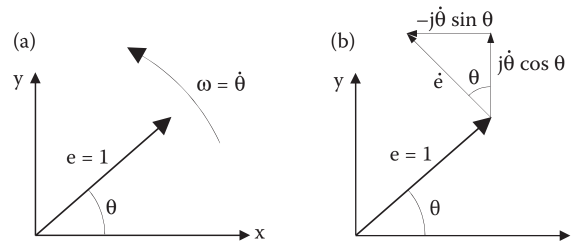


FIGURE 5.7 (a) Rotating unit vector and (b) the equivalent vector representation.

$$\dot{e} = \dot{\theta}(-i\ \cos\theta + j\sin\theta) \tag{5.25}$$

See Figure 5.7b:

$$|\dot{e}| = \dot{\theta}\sqrt{\sin^2\theta + \cos^2\theta} = \theta = \omega \tag{5.26}$$

$$e = \dot{\theta}k\ldots(\omega \text{ is in the z direction}) \tag{5.27}$$

$$\dot{e} = \omega xe\ldots\left(kxj = -i, kxi = j\right) \tag{5.28}$$

5.8 VECTOR OF POINT IN A ROTATING REFERENCE FRAME

In Figure 5.8, a point 'P' having a vector position 'r' is translating in the xy plane. 'e_1' and 'e_2' identify the position of the vector in respect to a reference frame that is rotating with an angular velocity 'ω'. 'r_1' and 'r_2' are components of the motion with respect to the 'e_1' and 'e_2' directions.

The position of the vector is expressed as follows:

$$r = r_1e_1 + r_2e_2 \tag{5.29}$$

The derivative with respect to time:

$$\dot{r} = \dot{r}_1e_1 + r_1\dot{e}_1 + \dot{r}_2e_2 + r_2\dot{e}_2 \tag{5.30}$$

$$r_1\dot{e}_1 = r_1\omega xe_1 = \omega xr_1e \tag{5.31}$$

$$r_2\dot{e}_2 = r_2\omega xe_2 = \omega xr_2e \tag{5.32}$$

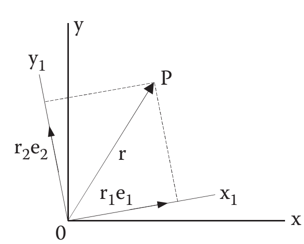


FIGURE 5.8 Vector of a point in rotating the reference frame.

From Equation 5.28

$$\frac{de}{dt} = \omega x e,$$

Therefore,

$$\dot{r} = \dot{r}_1 \dot{e}_1 + \dot{r}_2 e_2 + \omega x r_1 e_1 + \omega x r_2 e_2 \quad (5.33)$$

$$= \dot{r}_1 + \omega x (r_1 e_1 + r_2 e_2) \quad (5.34)$$

$$= \dot{r}_1 + \omega \text{ x } r \quad (5.35)$$

(r_1 = rate of change of 'r' as measured relative to e_1, e_2)

$$\dot{r}_1 = \dot{r}_1 e_1 + \dot{r}_2 e_2 \quad (5.36)$$

5.9 VELOCITY OF A POINT IN A MOVING REFERENCE FRAME

Consider a point 'P' moving within a reference frame 'xy' as shown in Figure 5.9 having a position vector of

$$r_p = r_{o'} + r \quad (5.37)$$

$r_{o'}$ locates o′ relative to the fixed reference frame xy axes and 'r' locates the point 'P' relative to axes (x_1, x_2).

$$r_p = r_{o'} + r \quad (5.38)$$

$$v_p = \dot{r}_p = \dot{r}_{o'} + \dot{r} \quad (5.39)$$

$$\dot{r} = \dot{r}_r + \omega \text{ x } r \quad (5.40)$$

$$\dot{r}_p = \dot{r}_{o'} + \dot{r}_r + w \text{ x } r \quad (5.41)$$

The moving frame is not rotating, $\omega = 0$

$$v_p = \dot{r}_p = \dot{r}_{o'} + r_r \quad (5.42)$$

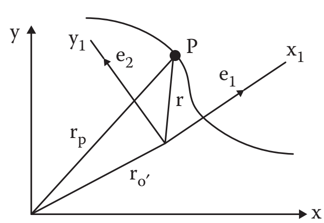


FIGURE 5.9 Vector of a point.

5.10 ACCELERATION OF A PARTICLE

Differentiating (with respect to time) the equation for velocity 'v' results in the equation for acceleration 'a'

$$a_p = \ddot{r}_p = \ddot{r}_{o'} + \ddot{r} \tag{5.43}$$

Following the principles outlined above

$$r = r_r + \omega \text{ x } r \tag{5.44}$$

$$\ddot{r} = \frac{d}{dt}(\dot{r}_r + \omega \text{ x } r) = \frac{d\dot{r}_r}{dt} + \frac{d}{dt}(\omega \text{ x } r) \tag{5.45}$$

$$\frac{d\dot{r}_r}{dt} = \ddot{r}_r + \omega \text{ x } \dot{r}_r \tag{5.46}$$

$$\frac{d}{dt}(\omega \text{ x } r) = \dot{\omega} \text{ x } r + \omega \text{ x } (\dot{r}_r + \omega \text{ x } r) \tag{5.47}$$

$$= \dot{\omega} \text{ x } r + \omega \text{ x } \dot{r}_r + \omega \text{ x } (\omega \text{ x } r) \tag{5.48}$$

Therefore,

$$r = \ddot{r}_r + \dot{\omega} \text{ x } r + \omega \text{x}(\omega \text{ x } r) + 2\omega \text{ x } \dot{r}_r \tag{5.49}$$

$$a_p = \ddot{r}_o + \ddot{r}_r + \dot{\omega} \text{ x } r + \omega \text{ x } (\omega \text{ x } r) + 2\omega \text{ x } \dot{r}_r \tag{5.50}$$

$$a_p = \ddot{r}_o + \ddot{r}_r \tag{5.51}$$

5.11 KINEMATICS OF RIGID BODIES IN ONE PLANE

The definition of a 'rigid body' is where all the internal points are fixed relative to each other. There are three co-ordinates required to identify the position and orientation of the body in plane motion. A rigid body will have three degrees of freedom in plane motion.

Further definitions:

- Rectilinear translation is where all the points comprising the rigid body move in a straight line.
- Curvilinear translation is where all the fixed points in the rigid body maintain the same orientation and remain fixed while the body moves along a curved path.
- Rotation about a fixed line, where all the points in a rigid body move in a circular motion around a fixed line.
- Plane motion, each point in a rigid body remains fixed to each other while moving in a path parallel to a fixed plane.

The equations for motion for a particle can be used to develop the plane motion of a rigid body. These are the same equations for the motion of a point, but as all the points are fixed relative to each other, the terms involving velocity and acceleration of these points relative to each other will be zero.

The equations for velocity and acceleration where the reference axes 'x' and 'y' are fixed to a rigid body are as follows:

From Equation 5.37:

$$r_p = r_{o'} + r$$

Point velocity:

$$v_p = v_{o'} + w \cdot r \quad \text{(as above but terms involving } \dot{r}_r \text{ are zero).}$$

Point acceleration:

$$a_p = a_{0'} + w \cdot r + w(w \cdot r) \quad \text{(as above but terms including } \dot{r}_r \text{ and } \ddot{r}_r \text{ are zero).}$$

If the position of the moving reference frame is fixed such that $r_{o'}$ is constant, then

$$v_p = \omega \cdot r \tag{5.52}$$

$$a_p = a \cdot r + \omega(\omega \cdot r) \tag{5.53}$$

5.12 INSTANTANEOUS CENTRE OF ROTATION

The instantaneous centre of rotation is also known as the 'instant centre of rotation'.

For a body moving in space, the motion can be defined from its position, velocity and acceleration of any point on the body. There is a point for which the instantaneous translational velocity is zero, which is only the rotation of the body about the point where it occurs. This is known as the 'instantaneous centre of rotation'. The relative values of the linear and angular velocities will determine its location. When the angular velocities are near zero (i.e., near translation motion), the location of the instantaneous centre of rotation will be near infinity.

When the location of a point and its associated velocity and angular velocity are known: (see Figure 5.10a and b). The point 'C' will be on a line passing through 'A', which is normal to the direction of the velocity as shown in Figure 5.10a:

$$r_A = \frac{v_A}{\omega} \tag{5.54}$$

If the location of two points and their respective velocities are known: where 'C' is the intersection of the lines drawn normal to the velocities of each of these points.

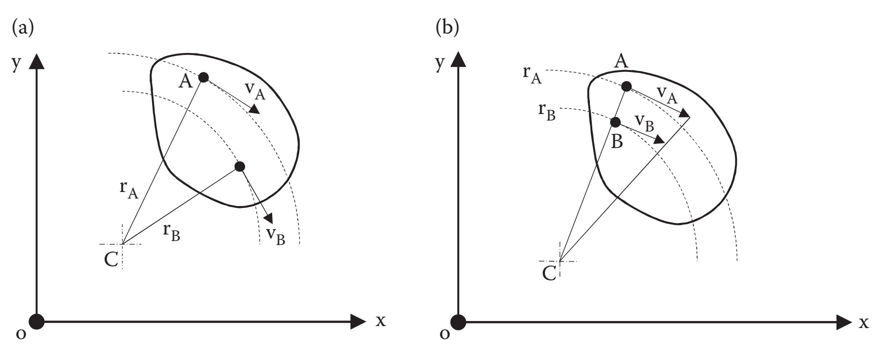


FIGURE 5.10 Instantaneous centre of rotation. (a) The location of two separate points and (b) coincident lines.

Where the lines are coincident, the radius can be established using geometry as in Figure 5.10b:

$$AC = \frac{AB(v_A)}{(v_A - v_B)} \tag{5.55}$$

5.13 KINEMATICS OF RIGID BODIES IN THREE DIMENSIONS

Three points are required to specify the position of a particle relative to the selected co-ordinate system and are identified as having three degrees of freedom. For a rigid body, the location of three separate points relative to a selected co-ordinate system is also required. As these points are relative to one another, only six independent co-ordinates are required to locate the body in a three-dimensional space and an unrestrained rigid body is therefore said to have six degrees of freedom. As an example, a rigid body can be positioned by locating the position of one of the bodies (three co-ordinates), then positioning a line on the body (two co-ordinates) and finally identifying a rotation about the line (one co-ordinate). This will sum up to six co-ordinates.

The motion of a rigid body existing in three dimensions can have a number of modes:

- *Rectilinear translation:* All points on a rigid body move in a straight line.
- *Curvilinear translation:* The relationship of all points in a rigid body remains fixed when the body moves along a curved path.
- *Rotation about a fixed axis:* The relationship of all points in a rigid body remains fixed when the body moves about a fixed axis.
- *Rotation about a fixed point:* All points in a rigid body maintain their fixed relationship with each other when the body moves in a circular motion about a fixed point.
- *General motion:* The rigid body motion includes translation and rotation.

Notes:

- If a rigid body rotates about the 'x' axis from position 'A' through an angle of $\pi/2$ and then proceeds to rotate about the 'y' axis, it will have moved to a certain position 'B'.
- If the same rigid body is rotated from position 'A' through an angle of $\pi/2$ about the 'y' axis and further rotated $\pi/2$ about the 'x' axis, it will have moved to position 'C'.
- Position 'C' will not be the same as position 'B'. Finite rotation does not obey the same laws as vector addition.

5.14 THEOREMS

1. *Euler's Theorem:* Two component rotations about different axes passing through a point are equivalent to a single resultant rotation about an axis passing through the point.
2. *Chasles's Theorem:* Any displacement of a rigid body may be compounded from a single rotation about any selected point plus a translation of that point.
3. *Poinsot's Central Axis Theorem:* Any finite displacement of a rigid body may be reduced to a single rotation about an axis plus translation parallel to the same axis. This theorem only relates to the displacement of the rigid body and not to paths taken by the points.

These theorems can be applied to angular velocities.

Any rotation of a rigid body can be described by a single angular velocity plus a translational velocity parallel to the angular velocity vector.

Any motion of a body about a point may be represented by a single velocity about an axis through that point.

Any motion of a rigid body may be represented by the velocity of a point plus the angular velocity about an axis passing through the points.

5.15 TRANSLATION MOTION

Rectilinear translation and curvilinear translation: all points on a body will move in parallel straight curved lines. There is no relative velocity or acceleration between any points on the body (see Figure 5.11).

Position:

$$r_P = r_Q + r_{PQ} \tag{5.56}$$

Velocity:

$$v_P = v_Q + v_{PQ} \tag{5.57}$$

Acceleration:

$$a_P = a_Q + a_{PQ} \tag{5.58}$$

5.16 ROTATION ABOUT A FIXED AXIS

Rotation of a rigid body about a fixed axis is shown in Figure 5.12. The vector of the rotary motion has sense and direction in accordance with the right-hand rule when this is aligned with the direction of the axis. There is zero velocity due to the rotation of the axis.

Velocity:

$$v_P = w \cdot r_P \tag{5.59}$$

Acceleration:

$$a_P = \dot{v}_P = \dot{\omega} \cdot r_P + \omega(\omega \cdot r_P) \tag{5.60}$$

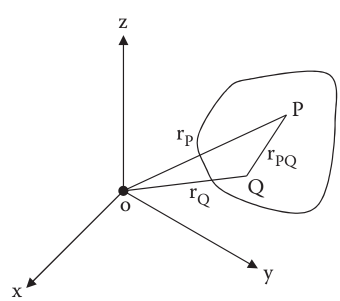


FIGURE 5.11 Rectilinear, translation and curvilinear.

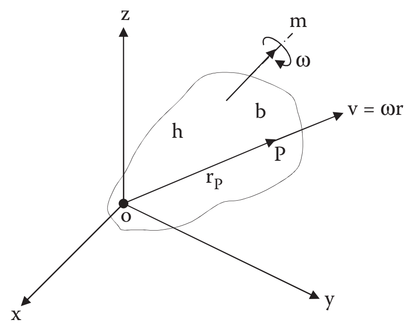


FIGURE 5.12 Rotation about a fixed axis.

5.17 ROTATION ABOUT A FIXED POINT

Figure 5.13 shows that the rotation of a rigid body about a fixed point can always be reduced to rotation of a body about an instantaneous axis of rotation.

Considering, for example, a rigid body (say a cone) which is rotating about a horizontal axis (say a shaft), angular velocity = ω_1 and the shaft is itself rotating about a vertical axis, with an angular velocity = ω_2.

If $\omega_2 = 0$, then the axis of rotation is the centre line of the shaft and the velocity of any point on the shaft is proportional to the radius (max = r) from the shaft. If the rotation about the vertical axis is increased to a certain value, the velocity of the shaft is proportional to ω_2 x the shaft length (max = l) from the axis radius. If the two angular velocities are the same and the radius of the cone (r = l), then at any instant the velocity of the top surface of the cone is zero, that is, the top surface is then the instantaneous axis of rotation.

This very simplified representation illustrates the principle; if ω_2 is increased, the cone representing the path of the instantaneous axis of rotation will be larger than the actual surface of the cone.

For the model illustrated, there is also a space cone, which is the path the instantaneous axis of rotation follows in space, that is, an inverted cone centred on the vertical axis.

For the example above, the instantaneous angular velocity of the cone will be

$$w = w_1 + w_2 \tag{5.61}$$

$$= w_1 i + w_2 j \tag{5.62}$$

In the real world, the motions and shapes are not the same as shown but the interactions between the body rotation on its axis, and the angular motion in space still result in an instantaneous axis of

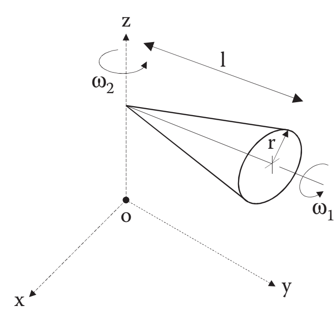


FIGURE 5.13 Rotation about a fixed axis.

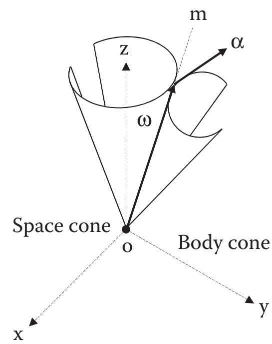


FIGURE 5.14 Rotation about a fixed point.

rotation between an instantaneous body cone and an instantaneous space cone. The body cone may rotate outside or inside the space cone (see Figure 5.14).

When determining the instantaneous angular velocity, the angular acceleration 'α' is tangential to the contact point of the two cones as shown in Figure 5.14. The velocity and acceleration at any point are simply determined as below.

Velocity:

$$v_P = \omega \cdot r \tag{5.63}$$

Acceleration:

$$a = \dot{v} = \dot{\omega} \cdot r + \omega(\omega \cdot r) \tag{5.64}$$

$$= \alpha \cdot r + \omega(\omega r) \tag{5.65}$$

5.18 GENERAL MOTION

The general case of 3D motion can be reduced to translation + rotation around a fixed axis. This is basically a generalisation of the theorems described above.

For a body possessing linear and angular motion, it is often not possible to have an instantaneous axis of rotation as all the points may have non-zero velocities. The most convenient method of kinematic analysis of rigid bodies in 3D space is to use the principles of relative motion.

The following reference system may be either a translating or a rotating one.

A translating reference system (see Figure 5.15):

This motion simply develops the motion already studied in previous sections for particle motions.

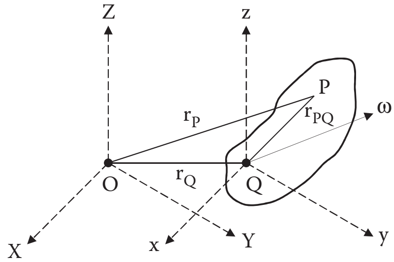


FIGURE 5.15 Translating a reference system.

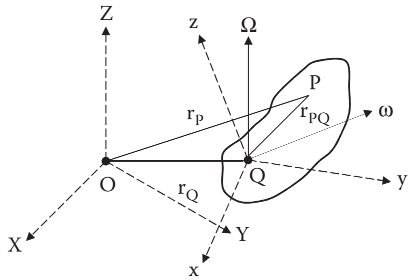


FIGURE 5.16 Rotating a reference system.

The basic motion equations are

$$r_P = r_Q + r \quad (5.66)$$

$$v_P = v_Q + v_{PQ} \quad (5.67)$$

$$a_P = a_Q + a_{PQ} \quad (5.68)$$

The above equations are sufficient if the angular velocity is zero but if this is not the case, the following equations for velocity and acceleration will be found to be more definitive.

Velocity:

$$v = v_Q + \omega \cdot r_{PQ} \quad (5.69)$$

Acceleration:

$$a_P = v_P = v_Q + \omega \cdot r_{PQ} + \omega(\omega \cdot r_{PQ}) \quad (5.70)$$

$$= a_Q + \alpha \cdot r_{PQ} + \omega(\omega \cdot r_{PQ}) \quad (5.71)$$

A rotating reference system (see Figure 5.16):

A more general form of the relative reference axis method using the rotating reference axes. The reference axes x, y, z are rotating with an angular velocity of Ω. The rigid body will have a rotation velocity 'ω' as shown before.

The basic motion equations are as shown above. The expressions for velocity and acceleration of point 'P' are shown below.

The derivation is a simple extension of that provided above for 2D motion with a rotating relative axis with the third dimension (z) added.

Velocity:

$$v_P = \dot{r}_Q + \Omega \cdot r_{PQ} + \dot{r}_r \quad (5.72)$$

$$= v_Q + \Omega \cdot r_{PQ} + v_r \quad (5.73)$$

Acceleration:

$$a_P = \dot{v}_P \quad (5.74)$$

$$= \dot{a}_Q + \dot{\Omega} \cdot r_{PQ} + \Omega(\Omega \cdot r_{PQ}) + 2\Omega \cdot v_r + a_r \quad (5.75)$$

v_r and a_r = the velocity and acceleration of 'P' relative to the rotating x, y, z axis.

Note: Equations 5.72 through 5.75 are based on the general case where the angular velocity of the rigid body (ω) is different from the angular velocity of the axis (Ω). If the reference axis is fixed to the body, then ω will be equal to Ω and v_r and a_r will be equal to zero. For this case, the formula will be the same as for the translating reference axis as above.

6 Mechanical Vibrations

6.1 INTRODUCTION

Mechanical vibrations are defined as oscillations in mechanical dynamic systems and are the motions of a particle or a body or system of connected bodies that have been displaced from a position of equilibrium. The majority of vibrations are undesirable in machines or structures as they can result in increased stresses, causing increased wear such as fretting and increased bearing loads. Mechanical fatigue can also result from vibrations and rotating machine parts including aero engine parts and will need careful balancing in order to prevent any damage resulting from vibrations.

Although most vibration problems are undesirable, such as the Tacoma Narrow Bridge failure in the United States in the 1940, and innumerable airframe failures resulting from vibration-induced fatigue, some mechanical systems such as the Beal free-piston Stirling engine rely on the vibration characteristics of the system to function correctly. In the mining and quarrying industries, these rely on sifting different sized particles using vibrating screed beds. In the manufacturing industry, vibration conveyors are used to convey components from one machining process to another.

Vibrations can be classified into four basic categories:

1. Free
2. Forced
3. Self-excited
4. Random

Free vibration of a system is vibration that occurs in the absence of any external force.

External force acting on a system will cause forced vibrations; in this instance, the exciting force is continuously supplying energy to the system. These types of vibrations may be either deterministic or random (see Figure 6.1).

Self-excited vibrations are periodic and deterministic oscillations. Under certain conditions, the equilibrium state becomes unstable and any disturbances will cause the perturbations to grow until some effect limits any further growth. This is in contrast to force vibrations, where the exiting force is independent of the vibrations and can still persist even when the system is prevented from vibrating.

This chapter will concentrate on most of the aspects of vibration, ranging from simple harmonic free vibrations to forced vibrations.

6.2 SINGLE DEGREE OF FREEDOM: FREE VIBRATIONS

6.2.1 Free Natural Vibrations

A free vibration is one that occurs naturally without any energy being added to the vibrating system. The vibration is started by an input of energy but the vibrations die away over time as the energy is dissipated. In each case, when the body is moved away from the rest position, there is a natural force that will try to restore the body back to its rest position. This can be better demonstrated by considering first a pendulum which can be considered as one degree of freedom, where the pendulum is displaced from its rest position and allowed to swing backwards and forwards with its amplitude gradually diminishing as the pendulum loses energy under the influence of gravity. This can be

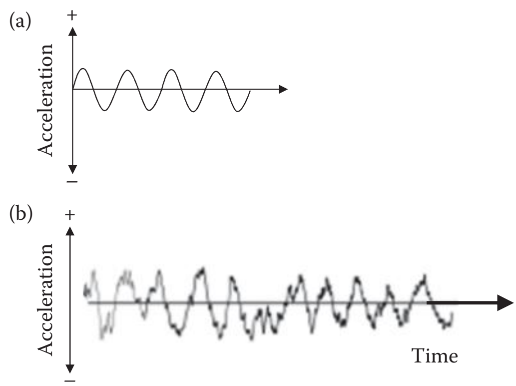


FIGURE 6.1 (a) Deterministic and (b) random vibrations.

further demonstrated when considering a mass suspended on a tension spring hanging vertically. With the mass displaced vertically downwards and the spring 'stretched', when the mass is released it will continue to rise and fall until again the energy in the spring is dissipated and the mass will come to rest in its equilibrium position (see Figure 6.2a and b).

The motion that these two examples display is known as 'simple harmonic motion' or generally referred to as 'SHM'.

This motion is characterised by the fact that when the displacement is plotted against time, the resulting graph is basically sinusoidal. The displacement can be either angular (e.g., the angle moved by the simple pendulum) or linear (e.g., the displacement of the mass on the spring). Although this study is concerned with natural vibrations, it may help to understand the nature of SHM if the forced vibration produced by a mechanism such as the Scotch yoke is considered.

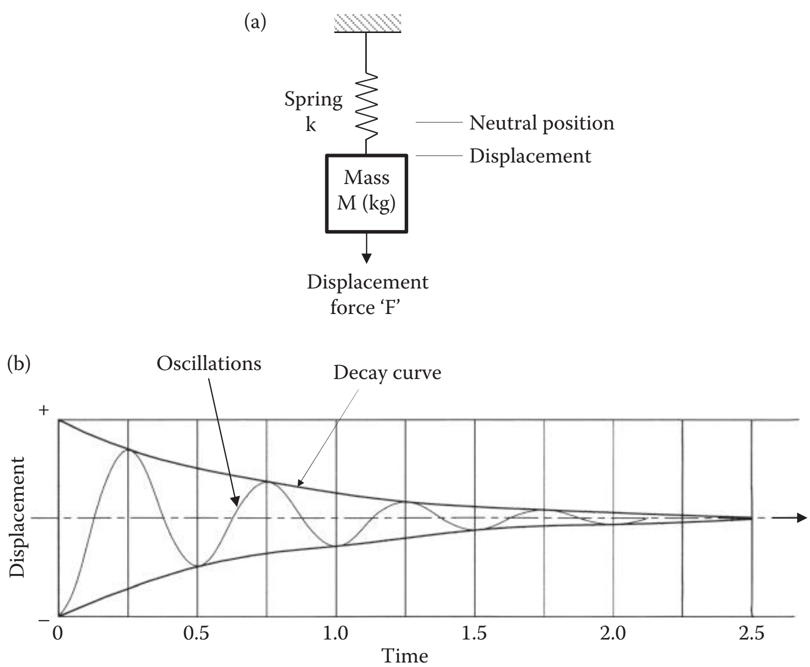


FIGURE 6.2 Deflected mass on a spring (a) and resultant vibration curve (b).

6.2.2 Simple Harmonic Motion

Consider Figure 6.3 which shows a typical Scotch yoke arrangement. With the crank rotating at a constant ω rad/s, a pin fitted in the crank slides in the slot of the yoke and point 'P' on the yoke will oscillate up and down as it is constrained to move only in the vertical direction by the bearing through which it is allowed to slide. The motion of point 'P' is considered to be sinusoidal when the displacement of the yoke against the angular position of the crank is plotted against time and therefore by definition is SHM. The point 'P' moving up and down will at any instantaneous point have a displacement 'x', a velocity 'v' and an acceleration 'a'.

With the pin located at radius 'R' from the centre of the crank, the vertical displacement of the pin from the horizontal centreline at any point is 'x'. This is also the displacement of point 'P'. The yoke reaches a maximum displacement equal to 'R' when the pin is at the top and '–R' when at the bottom. This is the amplitude of the oscillation. If the crank is rotating at a constant ω rad/s, then after time t s the angle rotated will be $\theta = \omega t$ radians. From the right-angled triangle (Figure 6.3b), $x = R \sin(\omega t)$ and the graph of x against θ is shown in Figure 6.4a.

Velocity is the rate of change of distance with respect to time and in calculus form

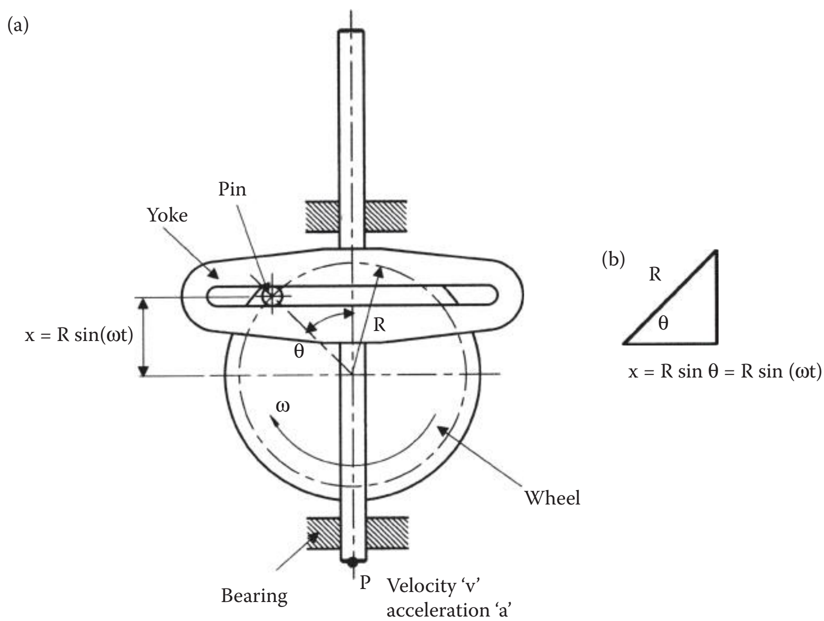
$$v = \frac{dx}{dt} \tag{6.1}$$

If 'x' is differentiated with respect to 't', the following result is found:

$$v = \frac{dx}{dt} = \omega R \cos(\omega t) \tag{6.2}$$

This plot is also shown in Figure 6.4b.

The maximum velocity of the yoke is ωR and occurs when the pin in the crank passes through the horizontal position. Positive (+) is considered vertically upwards and negative (–) vertically downwards.



FIGUE 6.3 (a) and (b): The Scotch yoke arrangement.

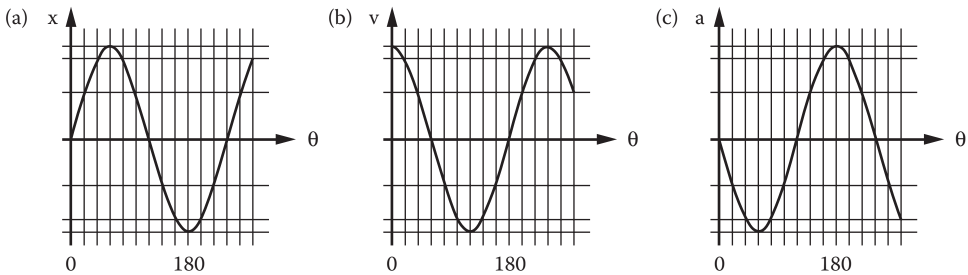


FIGURE 6.4 Graph of the Scotch yoke rotation. (a) Displacement, (b) velocity and (c) acceleration.

Acceleration is the rate of change of velocity with respect to time and again in calculus form:

$$a = \frac{dv}{dt} \tag{6.3}$$

Differentiating velocity 'v', the following is obtained

$$x = 20\sin\left((50 \times 0.3142) + \frac{\pi}{8}\right)$$

$$x = 20\sin\left(1.571 + \frac{\pi}{8}\right) \quad a = \frac{dv}{dt} = -\omega^2 R\sin(\omega t)$$

$$= 20\sin(1.9637)$$

$$= 18.481 \tag{6.4}$$

This plot is also shown in Figure 6.4c.

The amplitude is $\omega^2 R$ and this is positive at the bottom of the crank and negative at the top (when the yoke is about to change direction).

Now, since $R\sin(\omega t) = x$, substituting for x

$$a = -\omega^2 x \tag{6.5}$$

This is the usual definition of SHM and the equation states that any body that performs sinusoidal motion must have an acceleration that is directly proportional to the displacement and will always be directed to the point of zero displacement. The constant of proportionality is ω^2.

Any vibrating body that has a motion that can be described in this way must vibrate with SHM and have the same equations for displacement, velocity and acceleration.

6.2.2.1 Angular Frequency, Frequency and Periodic Time

The angular velocity of the crank is 'ω' but in any vibration problem such as the mass on a spring, this is referred to as the angular frequency as no physical crank exists.

The frequency of the crank in revolutions/s is equivalent to the frequency of the vibration. If the crank is rotating at 2 rev/s, the time for one revolution is 1/2 s. If the crank is rotating at 5 rev/s, the time for one revolution will be 1/5 s. Hence, if the crank is rotating at f rev/s, then the time for one complete revolution will be 1/f. This relationship is important and gives the periodic time.

Periodic time 'T' is the time required to complete one cycle.

Where f is the frequency or number of cycles per second, it follows that:

$$T = \frac{1}{f} \tag{6.6}$$

and

$$f = \frac{1}{T} \tag{6.7}$$

It can be seen that each cycle of an oscillation will be equivalent to one rotation of the crank and one revolution is an angle of 2π radians.

Therefore, when θ = 2π, t = T. Hence, it follows that since θ = ωt, 2π = ωT. Rearranging this will result in ω = 2π/T. Substituting T = 1/f, therefore,

$$\omega = 2\pi f \tag{6.8}$$

6.2.2.2 Equations for SHM

From the three equations derived previously

$$\text{Displacement: } x = R \sin(\omega t) \tag{6.9}$$

$$\text{Velocity: } v = \frac{dx}{dt} = \omega R \cos(\omega t) \tag{6.10}$$

$$\text{Acceleration: } a = \frac{dv}{dt} = -\omega^2 R \sin(\omega t) \tag{6.11}$$

The plots of x, v and a against the angle θ are shown in Figure 6.4a. In the calculations made so far, the measured angle from the horizontal position 'θ' was decided and concluded that the time was zero at this point. If the timing had been started after the angle had reached a value of 'φ' from this point, in this case 'φ' would be called the phase angle and be measured in radians.

The resulting equations for displacement, velocity and acceleration can then be rewritten as follows to take account of the phase angle

$$\text{Displacement: } x = R \sin(\omega t + \varphi) \tag{6.12}$$

$$\text{Velocity: } v = \frac{dx}{dt} = \omega R \cos(\omega t + \varphi) \tag{6.13}$$

$$\text{Acceleration: } a = \frac{dv}{dt} = -\omega^2 R \sin(\omega t + \varphi) \tag{6.14}$$

From Figure 6.4a and b, the plots of x, v and a are the same but from Figure 6.4b it will be noted that the vertical axis has been displaced by 'φ'. The point to note between Figure 6.4a and b is that the velocity curve has been displaced by 1/4 cycle (90°) to the left, and the acceleration curve has been displaced a further 1/4 cycle, making it 1/2 cycle out of phase with 'x'.

EXAMPLE 6.1

The displacement of a body subject to an SHM is described by the following equation

$$x = R\sin(\omega t + \varphi) \text{ (from Equation 6.12)}$$

where
R = amplitude
ω = natural frequency
φ = phase angle

Given that $R = 20$ mm, $\omega = 50$ rad/s and $\varphi = \pi/8$.
Calculate the following:

1. Frequency
2. Periodic time
3. Displacement, velocity and acceleration when $t = T/4$

Frequency:

$$\begin{aligned} f &= \frac{\omega}{2\pi} \\ &= \frac{50}{2\pi} \\ &= 7.9577 \text{ Hz} \end{aligned}$$

Periodic time:

$$\begin{aligned} T &= \frac{1}{f} \\ &= 0.12566 \text{ s} \end{aligned}$$

Time (t):

$$\begin{aligned} t &= \frac{T}{4} \\ &= 0.03142 \text{ s} \end{aligned}$$

Displacement (solve for $t = 0.03142$ s):

$$x = 20\sin\left((50 \times 0.3142) + \frac{\pi}{8}\right)$$

$$\begin{aligned} x &= 20\sin\left(1.571 + \frac{\pi}{8}\right) \\ &= 20\sin(1.9637) \\ &= 18.481 \text{ mm} \end{aligned}$$

The equations for velocity 'v' and acceleration 'a':

$$\begin{aligned} x &= 20\sin(\omega t + \varphi) \\ v &= 20\omega\cos(\omega t + \varphi) \\ &= 20 \times 50 \times \cos(1.9637) \\ &= -382.872 \text{ mm/s} \end{aligned}$$

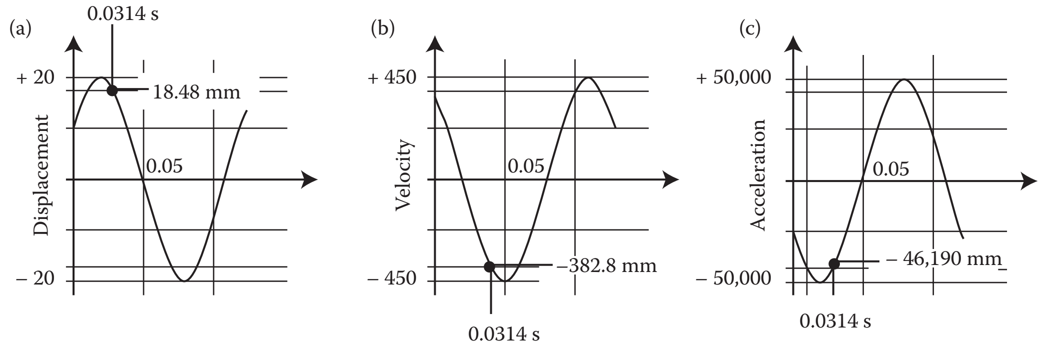


FIGURE 6.5 Figure for Example 6.1. (a) Displacement, (b) velocity and (c) acceleration.

$$\begin{aligned} a &= -20\omega^2 \sin(\omega t + j) \\ &= -20 \times 50^2 \sin(1.9637) \\ &= -46{,}190.061\,\text{mm/s}^2 \end{aligned}$$

These results are confirmed in Figure 6.5.

6.2.2.3 Free Natural Vibrations of a Single-Degree-of-Freedom System

In this section, it will be shown that some simple cases of natural vibrations are extended examples of SHM. One important point common to all these cases is that there must be a natural force that causes the body to move to the rest position. One further point that is common to all the following examples is that the body must possess a mass (inertia) and in order to accelerate this mass an inertial force or torque must be present.

6.2.2.3.1 Simple Pendulum

In this case, the restoring force is gravity. When the pendulum shown in Figure 6.6 is displaced through an angle ‘θ’, the weight will try to restore it to the rest position. This analysis is based on a moment of force (torque).

Note: Mass is indicated as ‘M’. This should not be confused with ‘m’, which is being used as an abbreviation for metre.

Restoring force:
Weight = Mg
Torque = T · g = weight × Mg(L sin θ)

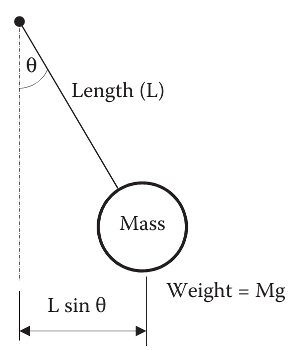


FIGURE 6.6 Simple pendulum.

Inertia torque:
As the pendulum has an angular acceleration 'α' as the pendulum slows down and then speeds up, it requires an inertial force to produce this effect. T_j denotes this torque.

From Newton's second law for angular motion, $T_j = I\alpha$, where α is the angular acceleration and I is the moment of inertia.

It is assumed that the mass is concentrated at radius 'L' from the pivot point and the moment of inertia is then $I = ML^2$.

Balance of moments:
If there is no applied torque from any external source acting on the mass, then the total torque acting on the body must equal zero, that is,

$$\begin{aligned} T &= \frac{1}{f} \\ &= 0.0498 \text{ s} \end{aligned}$$

The sine of small angles is very similar to the angle itself in radians. The smaller the angle, the more accurate it becomes. In such cases, sin(θ) = θ radians; therefore, this expression can be simplified to

$$\begin{aligned} g\alpha &= -L\alpha \\ \alpha &= -\left(\frac{g}{L}\right)\theta \end{aligned} \tag{6.15}$$

This expression meets the requirements for SHM since the acceleration 'α' is directly proportional to the displacement 'θ' and the minus sign indicates that the mass is always accelerating towards the rest position. It follows that the constant of proportionality is (g/L), that is,

$$\omega^2 = \frac{g}{L}, \quad \omega = \left(\frac{g}{L}\right)^{\frac{1}{2}} \tag{6.16}$$

If this displacement 'θ' is plotted against time 't', similar graphs as shown in Figure 6.4a and b will result. The displacement in this example is angle and is not to be confused with the angle on the Scotch yoke. The frequency of oscillation is obtained from

$$f = \frac{\omega}{2\pi}$$

Hence

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{6.17}$$

It will be noted that mass does not enter into the equation for frequency. On the earth, the only way to alter the frequency is to alter the length 'L'.

Please remember that the above equations are valid only if the pendulum swings through a small angle. If the angle is large, the motion will not be a perfect SHM.

EXAMPLE 6.2

A mass is suspended by a string 60 mm long. It is nudged so that it makes a small swinging oscillation. Determine the frequency and the periodic time of the swing.

$$f = \left(\frac{1}{2}\pi\right)\left(\frac{g}{L}\right)^{\frac{1}{2}}$$

$$f = \left(\frac{1}{2}\pi\right)\left(\frac{9.81}{0.06}\right)^{\frac{1}{2}}$$
$$= 20.085\ \text{Hz}$$

$$T = \frac{1}{f}$$
$$= 0.0498\ \text{s}$$

6.2.2.4 Elementary Parts of a Vibrating System

Up to now, a simplified introduction to SHM has been considered. Taking a more academic approach to a vibrating system that consists of a spring (a means of storing potential energy), a mass or inertia (a means of storing kinetic energy) and a damper (a means by which energy is gradually lost from the system), this is as shown in Figure 6.7. An undamped vibrating system will involve the transfer of potential energy to kinetic energy and kinetic energy back to potential energy.

In a damped vibrating system, some energy is dissipated in each cycle of vibration and will require to be replaced by an external force if a steady state of vibration is to be maintained.

6.2.2.5 Linear Elastic Oscillations

The majority of natural oscillations will occur due to the restoring force due to a spring. The spring can be considered to be an elastic body supporting a mass. The spring can be any structural member of a discrete length and does not have to be a helical spring (either tension or compression) in the literal sense. Although a helical spring is shown diagrammatically, it represents any elastic member that has a spring stiffness ‘k’ and will either extend or compress, twist or bend under a specific load.

Consider a mass suspended on a spring as shown in Figure 6.8. The mass is subject to a force ‘F’ which displaces the mass downwards. The spring is extended by a distance ‘x_o’ and this is called the initial displacement.

If the force is removed, the mass will begin to oscillate up and down with an SHM.

Let F_s = spring force that tries to return the mass to its original rest position. From spring theory

$$F_s = kx$$

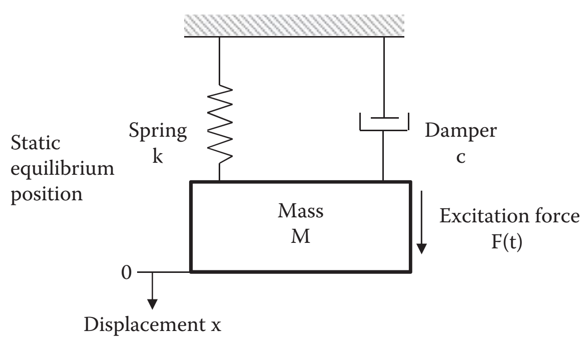


FIGURE 6.7 Elementary parts of a vibrating system.

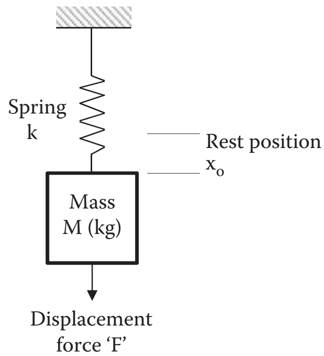


FIGURE 6.8 Mass suspended on a spring.

As the motion of the mass has an acceleration when released, an inertial force 'F_i' is present. From Newton's second law of motion

$$F_i = Ma \text{ (mass} \times \text{acceleration)}$$

Therefore, balancing the forces acting on the mass gives

$$F = F_i + F_s = Ma + kx$$

If the mass is physically disturbed and released so that it is oscillating freely, the applied force is obviously zero and this is the requirement for oscillation to be free and natural; hence,

$$0 = ma + kx,$$

rearranging the equation for 'a':

$$a = -\left(\frac{k}{M}\right) \cdot x \tag{6.18}$$

Thus, this is the equation for SHM where the acceleration is directly proportional to the displacement and is directed towards the rest position of the mass.

If the displacement 'x' is plotted against time 't', a sinusoidal graph would result. The constant of proportionality k/M is the square of the angular frequency so

$$\omega = \sqrt{\frac{k}{M}} \tag{6.19}$$

The frequency of oscillation will be

$$\begin{aligned} f &= \frac{\omega}{2\pi} \\ &= \frac{1}{2\pi}\sqrt{\frac{k}{M}} \end{aligned} \tag{6.20}$$

As this is a natural oscillation, the frequency is generally denoted as ω_n and f_n. This equation is true for all elastic oscillations.

EXAMPLE 6.3

A spring having a stiffness of 20 kN/m supports a mass of 4 kg. The mass is displaced 8 mm vertically downwards and then released to produce linear oscillations.

Calculate the frequency and the periodic time and the displacement, velocity and acceleration 0.05 s after the mass has been released.

$$\omega_n = \sqrt{\frac{k}{M}}$$
$$= \sqrt{\frac{20 \times 10^3 \text{ N/m}}{4 \text{ kg}}}$$
$$= 70.71 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi}$$
$$= 11.25 \text{ Hz}$$

$$T = \frac{1}{f_n}$$
$$= 0.089 \text{ s}$$

The oscillations start at the bottom of the cycle so $x_o = -8.0$ mm and the resulting graph of x against time will therefore be a negative cosine curve with an amplitude of 8.0 mm.

The equations that describe the motion are as follows:

$$x = x_o \cos \omega t \text{ (when } t = 0.05 \text{ s}, x = -8.0 \cos (70.71 \times 0.05))$$
$$x = 7.387 \text{ mm (Note: Angles are expressed in radians)}$$

To obtain the expression for velocity, the expression displacement is differentiated once

$$v = -\omega \cdot x_o \cdot \sin \omega t$$
$$v = -70.71\,(-8.0) \sin (70.71 \times 0.05)$$
$$= -217 \text{ mm/s}$$

Differentiating the displacement equation a second time will give the equation for acceleration, that is,

$$a = -\omega^2 x_o \cos \omega t \quad \text{since } x = x_o \cos \omega t \ a = -\omega^2 x$$
$$a = -70.712 \times 7.387$$
$$= -36{,}934.3 \text{ mm/s}^2$$

Figure 6.9a through c confirms these results.

6.2.2.6 Transverse Vibrations

A transverse vibration is where the motion is normal to the length. This generally occurs in beams and shafts and may be solved by either an angular or linear motion. In this section, a linear motion is used.

6.2.2.6.1 Cantilever

Consider a mass at the end of a cantilever beam as in Figure 6.10. If the mass is displaced at right angles to the beam, the beam will bend and using the beam deflection equations, it will be found

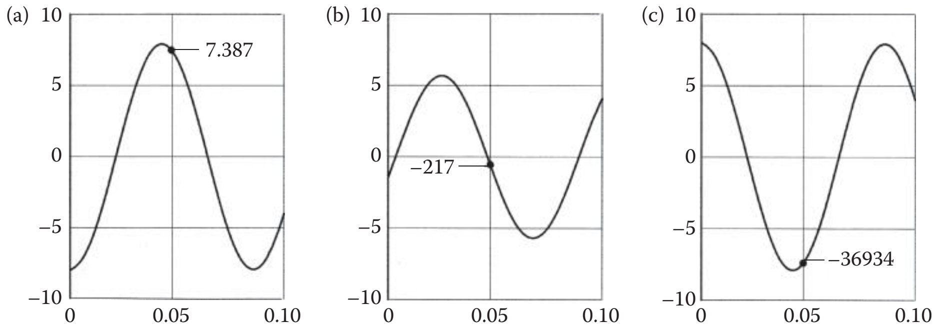


FIGURE 6.9 (a), (b), and (c): Example 6.3.

that the force is directly proportional to displacement for small beam deflections. If the beam deflections become large, then the constant of proportionality will fail.

The cantilever can be considered as a simple transverse spring.

The stiffness of the beam can be calculated using the beam theory where the deflection of the cantilever due to a point load acting at the end of the beam is given by

$$y = \frac{Fl^3}{3EI} \tag{6.21}$$

In a beam equation, 'y' is used to denote the deflection of the beam since 'x' has already been used. In the studies, thus far 'x' has been used to denote the deflection; therefore, 'x' will be used for future deflections.

Therefore, Equation 6.21 will be rewritten as

$$x = \frac{Fl^3}{3EI} \tag{6.22}$$

Hence, the stiffness will be

$$\begin{aligned} k &= \frac{F}{x} \\ &= \frac{3EI}{l^3} \end{aligned} \tag{6.23}$$

It can be shown that the theory is the same as for a mass on the end of a spring:

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{M}} \tag{6.24}$$

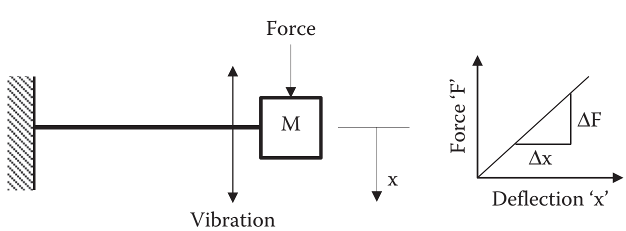


FIGURE 6.10 Cantilever beam with a mass at the free end.

$$= \frac{1}{2\pi}\sqrt{\frac{3EI}{Ml^3}} \tag{6.25}$$

6.2.2.6.2 Simply Supported Beam

Evaluating the deflection of a simply supported beam with a point load acting mid-span, the central deflection will be (see Figure 6.10)

$$x = \frac{Fl^3}{48EI} \tag{6.26}$$

and it follows that:

$$k = \frac{48EI}{l^3} \tag{6.27}$$

and

$$f_n = \frac{1}{2\pi}\sqrt{\frac{48EI}{Ml^3}} \tag{6.28}$$

6.2.2.6.3 Static Deflection

The stiffness can also be found by measuring the static deflection of the beam. Assuming that the mass deflects a small distance 'x_s' under its own weight, the force in this case is the weight; hence,

$$F = Mg \tag{6.29}$$

It follows that:

$$k = \frac{Mg}{x_s} \tag{6.30}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{M}} \tag{6.31}$$

and

$$= \frac{1}{2\pi}\sqrt{\frac{g}{x_s}} \tag{6.32}$$

This formula will work both for a cantilever and a simply supported beam. It may be noted that none of the foregoing calculations have considered the mass of the beam or shaft and provided this is small in comparison to the concentrated mass, the formula will give an accurate answer.

Later work will consider beams and shafts where the mass of the beam and shaft is significant in comparison with the load.

EXAMPLE 6.4

Consider a rod 20.0 mm in diameter and 1200 mm long is rigidly fixed at one end and having a mass of 2 kg acting at the free end. Ignoring the weight of the rod, calculate the frequency of transverse vibrations. Take E = 200 GPa.

Second moment of area for the circular section

$$I = \frac{\pi . D^4}{64}$$
$$= \frac{\pi \times 0.024}{64}$$
$$= 7.854 \times 10^{-9} \text{ m}^4 \quad (6.33)$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{3EI}{Ml^3}}$$
$$= \frac{1}{2\pi}\sqrt{\frac{3 \times 200 \times 10^9 \times 7.854 \times 10^{-9}}{2 \times 1.23}}$$
$$= 5.88 \text{ Hz} \quad (6.34)$$

EXAMPLE 6.5

A horizontal shaft sitting in simple bearings has a mass of 40 kg placed at mid-span. There is a deflection of 1.00 mm due to the mass. Ignoring the mass of the shaft, determine the frequency of transverse oscillation ($g = 9.81$ m/s^2).

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{x_n}}$$
$$= \frac{1}{2\pi}\sqrt{\frac{9.81}{0.001}}$$
$$= 15.76 \text{ Hz}$$

6.2.2.7 Energy Methods (Rayleigh)

A method for solving complex oscillations for a mass–spring system was developed by Rayleigh and is based on that. During an oscillation, the maximum kinetic energy of the oscillating mass is equal to the maximum strain energy. This is best illustrated by referring to Figure 6.11.

Let the maximum deflection of the mass be x_o.

The spring force will be $F = kx_o$.

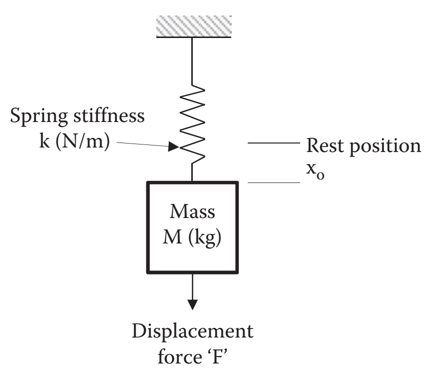


FIGURE 6.11 Rayleigh's method for transverse oscillations.

$$\text{The work done} = \frac{1}{2} F\, x_o$$
$$= \frac{1}{2} k\, x_o^2 \quad (6.35)$$

Let the displacement at any time be

$$x = x_o \sin \omega t \quad (6.36)$$

The velocity will be

$$v = \omega x_o \cos \omega t \quad (6.37)$$

The maximum velocity will be $\omega \cdot x_o$.

$$\text{The maximum kinetic energy} = \frac{1}{2} M v^2 \quad (6.38)$$

$$KE_{max} = \frac{1}{2} M \omega^2 x_o^2 \quad (6.39)$$

Equating energy and $(1/2)M\omega^2 x_o^2$

$$\omega^2 = \frac{k}{M} \quad (6.40)$$

It is important to point out that this result is independent of the deflection (x_o). This method is particularly useful for determining the frequency of transverse oscillations.

EXAMPLE 6.6

Consider the shaft depicted in Figure 6.12.

It is required to establish the natural frequency of the shaft, ignoring the mass of the shaft.

Take the flexural stiffness as $EI = 20{,}300\ N \cdot m^2$.

The solution to this problem can be derived using two different methods:

1. Static deflection method
2. Strain energy method

In this instance, the strain energy method will be used for the solution.

Determine the reactions R_A and R_B.

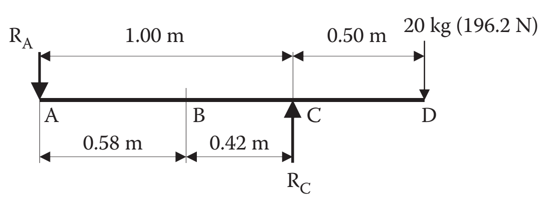


FIGURE 6.12 Shaft in Example 6.6.

Taking moments about R_A:

$1.5 \times 196.2 = 1.0 \times R_C$
$R_C = 294.3$ N (down).

Taking moments about R_C:

$1.00\ R_A = 0.50 \times 196.2$
$R_C = -98.1$ N (up).

Strain energy for section 'A' to 'C':

$$
\begin{aligned}
U &= \frac{1}{2EI}\int M^2\,dx \\
&= \frac{1}{2 \times 20{,}300}\int_0^1 (-98.1x)^2\,dx \\
&= \frac{98.1^2}{2 \times 20{,}300}\left[\frac{x^3}{3}\right]_0^1 \\
&= 0.079\ \text{J}
\end{aligned}
\tag{6.41}
$$

Strain energy for section 'C' to 'B':
This will be simplified if 'x' is measured from the free end where M = 196.2 x.

$$
\begin{aligned}
U &= \frac{1}{2EI}\int M^2\,dx \\
&= \frac{1}{2 \times 20{,}300}\int_0^{0.51} (-196.2x)^2\,dx \\
&= \frac{196.2^2}{2 \times 20{,}300}\left[\frac{x^3}{3}\right]_0^{0.5} \\
&= 0.0395\ \text{J}
\end{aligned}
$$

The total strain energy 'U' = 0.079 + 0.0395 = 0.1185 J.
Let the deflection produced at the free end be y_m.
The strain energy

$$
\begin{aligned}
U &= \frac{1}{2}F\,y_m \\
F &= Mg \\
&= 20\ \text{kg} \times g \\
&= 196.2\text{N}
\end{aligned}
\tag{6.42}
$$

Hence,

$$
\begin{aligned}
y_m &= 2 \times \frac{0.1185}{196.2} \\
&= 0.0012\ \text{m}
\end{aligned}
$$

Now if the mass is oscillating up and down sinusoidally

$$y = y_m \sin \omega t \tag{6.43}$$

The velocity of the oscillation will be

$$v = \omega\, y_m \cos(\omega t) \quad (6.44)$$

The kinetic energy will be

$$KE = \frac{1}{2}\frac{Mv^2}{2} \quad (6.45)$$

$$= \frac{1}{2}M\left\{\omega\, y_m \cos(\omega t)\right\}^2 \quad (6.46)$$

The maximum value of KE can be calculated as follows:
Now y_m is the deflection at the free end of the shaft.

$$= 0.0021 \text{ m}, \; M = 20 \text{ kg}$$

Equating energies

$$\frac{1}{2}M\{\omega\, y_m\}^2 = 0.1185$$

$$10\,\omega^2(0.0021^2) = 0.1185$$

$$\omega^2 = \frac{0.1185}{\{10(0.0021^2)\}} = 8229$$

$$\omega = 90.71 \text{ rad/ s}$$

$$f_n = \frac{90.71}{(2\pi)} = 14.43 \text{ Hz}$$

The static deflection of the shaft could have been determined using the beam theory.

6.3 DAMPED VIBRATIONS

This section covers the theory of natural vibrations with damping. In Section 6.2, it was intimated that once a system was set to vibrate naturally, it would carry on vibrating as the energy that had been put into the system by the initial disturbance has no way of escaping the system. In practice, the vibrations will gradually reduce in amplitude and die away over a period of time. Figure 6.13 shows the response of a system to varying degrees of damping following an initial displacement 'A'.

The definition of damping is where the free vibration of a system is controlled. A system that is critically damped returns to its stable reference position as quickly as possible without any under- or overshoot. A good example is the machine gun when the recoil mechanism of a gun is designed with critical damping such that the system returns to its firing position in the quickest time without any overshoot. A further example is the automotive suspension system where the vehicle passes over an obstacle and returns the vehicle to its correct ride height as quickly as possible without any undesirable undershoot.

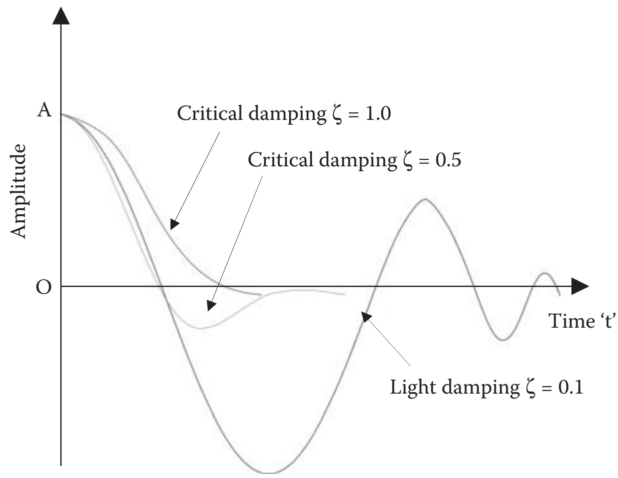


FIGURE 6.13 Damping following an initial displacement 'A'.

Damping can be accomplished by a number of methods including

1. Viscous damping
2. Coulomb damping (dry friction)
3. Inertial damping
4. Internal damping

The basic nomenclature used in damping is covered in Table 6.1. Table 6.2 tabulates the natural frequencies and coefficients 'K' for various modes.

6.3.1 Viscous Damping

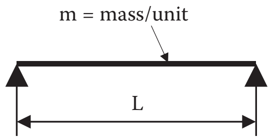
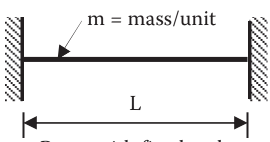
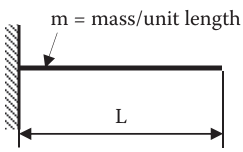
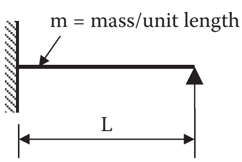
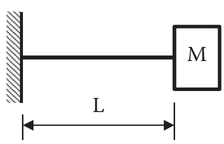
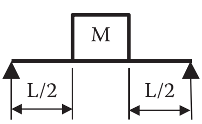
Damping that uses a fluid to provide the damping force is known as viscous damping.

Figure 6.14a and b shows two types of viscous dampers. These kinds of dampers are also known as dashpots. In Figure 6.14a, the dashpot uses a fluid (usually oil) and the piston has a number of small orifices passing through the piston. These holes are sized so that the fluid is restricted when it

TABLE 6.1
Nomenclature Used in Forced Vibrations

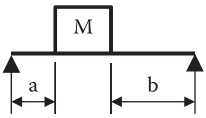
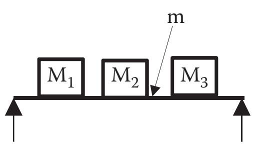
Symbol	Description	Units
L =	Length	m
m =	Mass per unit length	kg/m
I =	Area moment of inertia	m^4
g =	Acceleration due to gravity	(9.81 m/s^2) m/s^2
E =	Modulus of elasticity	N/m^2
G =	Torsional modulus	N/m^2
f =	Frequency of vibration	Hz
y =	Deflection	m
k =	Radius of gyration	m
J =	Shaft polar moment of inertia	m^4
s =	Shaft torsional stiffness (GJ/L)	Nm · rad

TABLE 6.2
Natural Frequencies and Coefficients 'K' for Various Modes

Case	Graphic	Equation	Harmonic Mode				
1	 Simply supported beam	$f = \frac{K}{2\cdot\pi\cdot L^2}\sqrt{\frac{E\cdot I}{m}}$	Mode	1	2	3	4
			K	9.87	39.5	88.8	158
2	 Beam with fixed ends	$f = \frac{K}{2\cdot\pi\cdot L^2}\sqrt{\frac{E\cdot I}{m}}$	Mode	1	2	3	4
			K	22.4	61.7	121.0	200.0
3	 Cantilever subject to transverse vibrations	$f = \frac{K}{2\cdot\pi\cdot L^2}\sqrt{\frac{E\cdot I}{m}}$	Mode	1	2	3	4
			K	3.52	22.0	61.7	121
4	 Cantilever one end fixed and simply supported at the free end	$f = \frac{K}{2\cdot\pi\cdot L^2}\sqrt{\frac{E\cdot I}{m}}$	Mode	1	2	3	4
			K	15.4	50.0	104.0	178.0
5	 Cantilever with mass at the free end (assume the beam has a negligible mass)	$f = \frac{1}{2\cdot\pi\cdot}\sqrt{\frac{3\cdot E\cdot I}{M\cdot L^3}}$					
6	 Central mass on simply supported beam (assume the beam has a negligible mass)	$f = \frac{1}{2\cdot\pi\cdot}\sqrt{\frac{48\cdot E\cdot I}{M\cdot L^3}}$					

continued

TABLE 6.2 (continued)
Natural Frequencies and Coefficients 'K' for Various Modes

Case	Graphic	Equation	Harmonic Mode
7	 Off-centre mass on a simply supported beam (assume the beam has a negligible mass)	$f = \frac{1}{2 \cdot \pi}\sqrt{\frac{3 \cdot E \cdot I \cdot L}{M \cdot a^2 \cdot b^2}}$	L = a + b
8	 Beam with multiple masses	$\frac{1}{f} = \frac{1}{f^2} + \frac{1}{f_1^2} + \frac{1}{f_2^2} + \frac{1}{f_3^2}$	Using Dunkerly's method of combined loading

passes from one side to the other. Because of this restriction, a force is required to move the piston within the dashpot and the force can be made to oppose any external force generated by the vibration.

In Figure 6.14b, air is the fluid and as the piston moves within the dashpot, it will suck or force air through the orifice in the bottom of the dashpot. Again, the orifice is sized to provide a resistance to the air as it passes through the orifice and this will provide a force opposing the motion of the piston.

It can be shown that for both systems the force opposing the motion (the damping force) is proportional to the velocity of the piston within the dashpot. The equation for this force is

$$\begin{aligned} F_{dashpot} &= \text{constant} \times \text{velocity} \\ &= c\,\frac{dx}{dt} \end{aligned} \tag{6.47}$$

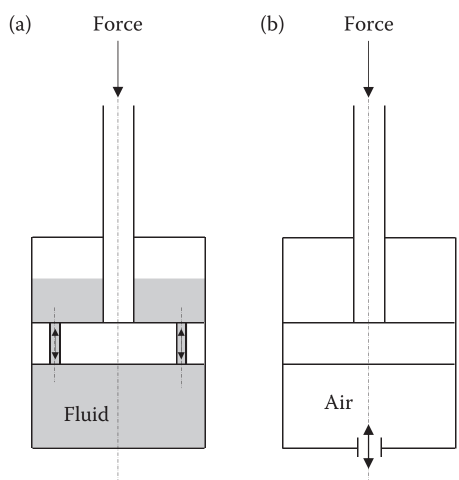


FIGURE 6.14 Dashpot designs. (a) Fluid dashpot and (b) pneumatic dashpot.

The constant of proportionality ‘c’ is called the damping coefficient and has units of Ns/m and is determined by the sizes of the orifices in the piston or the dashpot cylinder.

The damped system is characterised as in Figure 6.15.

For a free natural vibration with damping, the equations of motion for the above damper/spring/mass system with damping will be

$$M \cdot \frac{d^2x}{dt^2} = -k \cdot x - c \cdot \frac{dx}{dt} \tag{6.48}$$

Dividing Equation 6.48 through by M gives

$$\frac{d^2x}{dt^2} + \left(\frac{c}{M}\right) \cdot \frac{dx}{dt} + \left(\frac{k}{M}\right) \cdot x = 0 \tag{6.49}$$

Substituting ω_n^2 for k/M, making

$$\begin{aligned} \delta &= \frac{c}{2 \cdot M \cdot \omega_n} \\ &= \frac{c}{2 \cdot M \cdot \sqrt{k/M}} \end{aligned} \tag{6.50}$$

The following version of the equation will result:

$$\frac{d^2x}{dt^2} + 2 \cdot \delta \cdot \omega_n \cdot \frac{dx}{dt} + \omega_n{}^2 \cdot x = 0 \tag{6.51}$$

Now

$$\begin{aligned} 2\delta\omega_n &= 2 \cdot \frac{c}{2\sqrt{M \cdot k}} \sqrt{\frac{k}{M}} \\ &= \frac{c}{M} \end{aligned} \tag{6.52}$$

This equation may be solved by assuming a solution in the form $x = e^{s \cdot t}$ (s = constant).

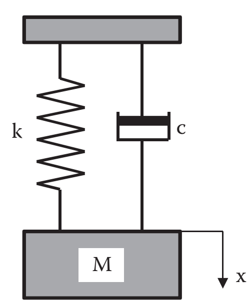


FIGURE 6.15 Spring–mass–viscous damper system.

Substituting this relationship into the equation will result in the following formula

$$\left[s^2 + \left(\frac{c}{M}\right)\cdot s + \left(\frac{k}{M}\right)\right]e^{st} = (s^2 + 2\delta\omega_n s + \omega_n^2)e^{st} = 0 \tag{6.53}$$

Hence,

$$s^2 + \left(\frac{c}{M}\right)\cdot s + \left(\frac{k}{M}\right) = s^2 + 2\delta\omega_n s + \omega_n^2 \tag{6.54}$$

There are two roots to this equation

$$s_{1.2} = -\frac{c}{2M} \pm \sqrt{\left(\frac{c}{2M}\right)^2 - \frac{k}{M}} = -\delta\omega_n \pm \sqrt{\delta^2\omega_n^2 - \omega_n^2}$$

$$= \omega_n\left(-\delta \pm \sqrt{\delta^2 - 1}\right) \tag{6.55}$$

The solution of this equation in general forms

$$x = Ae^{s_1}t + Be^{s_2}t \tag{6.56}$$

A and B are constants which can be evaluated from initial values of 'x' and dx/dt. The substitution of the roots into the general form will result in the following equation

$$x = e^{-\left(\frac{c}{2M}\right)t}\left(Ae^{\left(\sqrt{\left(\frac{c}{2M}\right)^2 - \frac{k}{M}}\right)t} + Be^{-\left(\sqrt{\left(\frac{c}{2M}\right)^2 - \frac{k}{M}}\right)t}\right) \tag{6.57}$$

Alternatively,

$$x = e^{-\delta\omega_n t}\left(Ae^{+\omega_n\left(\sqrt{\delta^2 - 1}\right)t} + Be^{-\omega_n\left(\sqrt{\delta^2 - 1}\right)t}\right) \tag{6.58}$$

The term $e^{-(c/2M)t}$ represents an exponential decaying factor. There are three general results arising from the expression within the brackets that will have a significant effect on the results, that is,

1. If $(c/2M)^2 = (k/M)\cdots(\delta^2 = 1)$, the factor within the bracket is 0 and the solution will be considered to be critically damped.
2. If $(c/2M)^2 < (k/M)\cdots(\delta^2 < 1)$, the factor inside the bracket is negative; therefore, the solution will be underdamped.
3. If $(c/2M)^2 > (k/M)\cdots(\delta^2 > 1)$, the factor inside the bracket is positive; therefore, the solution will be overdamped.
4. If $(\chi/2M)^2 = 0\cdots(\delta^2 > 0)$, the factor inside the bracket shows that the system is undamped.

6.3.2 Coulomb Damping

When a machine slide is accelerated along a slideway, friction will offer a resistance to the slide's movement. This resistance is referred to as Coulomb damping and is proportional to the friction in the system (see Figures 6.16 and 6.17).

Static friction occurs when the two objects are stationary or undergoing no relative motion. The frictional force 'F' exerted between the two surfaces having no relative movement cannot exceed a value that is proportional to the product of the normal force 'N' and the coefficient of static friction μ_s.

$$F_s = \mu_s \cdot N \tag{6.59}$$

Kinetic friction occurs when the two contacting surfaces are undergoing relative motion and sliding against each other. In this case, the friction force 'F' between the two surfaces is proportional to the product of the normal force 'N' and the coefficient of kinetic friction μ_k.

$$F_k = \mu_k \cdot N \tag{6.60}$$

In both these cases, the frictional force will always oppose the direction of movement between the surfaces. The normal force is perpendicular to the direction of motion and is equal to the weight (Mg) of the sliding object, where the damping force $F_d = \mu_{s.k} \cdot N$ and is independent of velocity.

The system is given an initial displacement 'A' and is then released as shown in Figure 6.13.

$$\frac{2\pi}{\omega_n}$$

The equations of motion for each direction

$$M\ddot{x} + kx \pm F_d = 0$$

$$x = \left(A \pm \frac{F}{k}\right)\cos \omega_n t \mp \frac{F}{k} \quad \mp \text{ Since 'F' opposes motion}$$

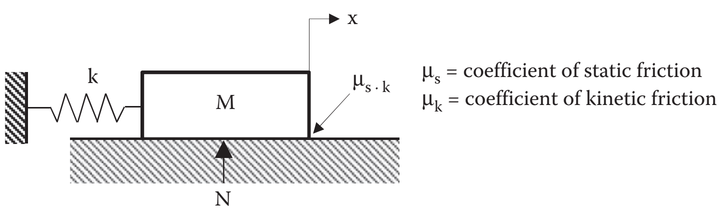


FIGURE 6.16 Coulomb damping.

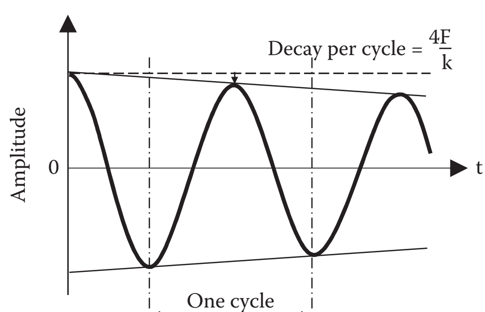


FIGURE 6.17 Coulomb diagram.

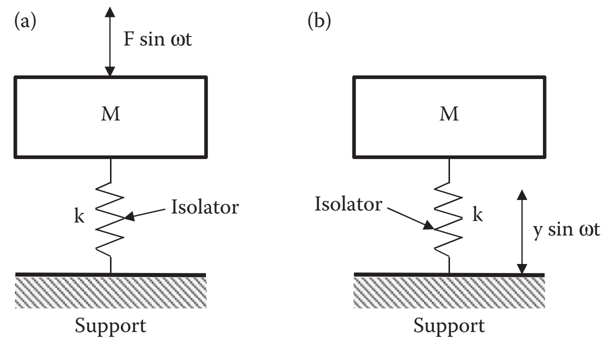


FIGURE 6.18 Mass subject to a vertical periodic force. (a) Active isolation and (b) passive isolation.

6.3.3 Inertial Damping

A significant mass is attached to the vibrating system. This has the effect of absorbing the vibration energy; it does not eliminate the vibration but introduces a phase lag such that when the vibration peak passes, the inertia of the mass absorbs that energy and this then reduces the instantaneous energy. An example is the counterweights fitted to the crankshaft of an engine as they even out the vibrations created within the engine.

A further example is the ballast fitted to boats where they damp out the rolling action created by the sea waves.

Internal damping (also known as hysteretic and viscoelastic damping) is mainly concerned with the support of a mass using an isolator between the mass and supporting structure such as the ground.

There are two methods of isolation:

1. Active isolation
2. Passive isolation

Figure 6.18a and b is a schematic diagram of a mass subjected to a vertical periodic force.

6.3.4 Internal Damping

In general, rubber mounts are used to isolate vibrations under items such as machine tools and so on.

D = Specific damping energy
= energy dissipated per cycle per unit volume of material
= area of stress–strain loop (Hysteresis loop)
U = Maximum energy stored per unit volume during loading
ψ = Specific damping capacity
α = Loss angle, the phase difference between stress σ and strain ε
$\tan \phi$ = Loss factor

$D = \pi\sigma\varepsilon \sin \phi$
$U = 1/2\ \sigma\varepsilon \cos \phi$
$\psi = D/(2U) = \pi \tan \phi$

For a mass supported by rubber isolators

$$\delta = \frac{1}{2} \tan \alpha$$

The value of α for natural rubber will vary between 4° and 11°.

For neoprene, α = 9° and for butyl α = 15°.

The equations of motion will be

$$x = A^{e-\delta\omega_n t} \sin(\omega_d t + \psi) \quad x + \tan\alpha\omega_n x + \omega_n^2 x = 0$$

with the solution

$$x = A^{e-\delta\omega_n t} \sin(\omega_d t + \psi)$$

6.4 SINGLE DEGREE OF FREEDOM: FORCED VIBRATIONS

In the previous section covering damped oscillations, it was shown that a free vibration will eventually die away over a period of time as the energy is dissipated by the damping. It was shown that the equation for the displacement of a damped oscillation is given by

$$x = Ce^{-\delta\omega_n t} \cos(\omega_n t) \tag{6.61}$$

where

δ is the damping factor

ω_n is the natural frequency

The following cases are described:

1. When δ > 1, the system is overdamped.
2. When δ = 1, the system is critically damped.
3. When δ < 1, the damped oscillations will die away with time.
4. When δ = 0, the system has no damping and a steady oscillation will occur.

Figure 6.19 illustrates the effect of the damping ratio.

If the damping factor 'δ' is less than zero, that is, negative, this would be the opposite to damping and would reverse the energy flow in that energy would be put back into the system instead of removing it. As the energy is put back into the system, it is possible for the oscillations to increase, the energy being supplied by an external force, and such oscillations are called forced oscillations.

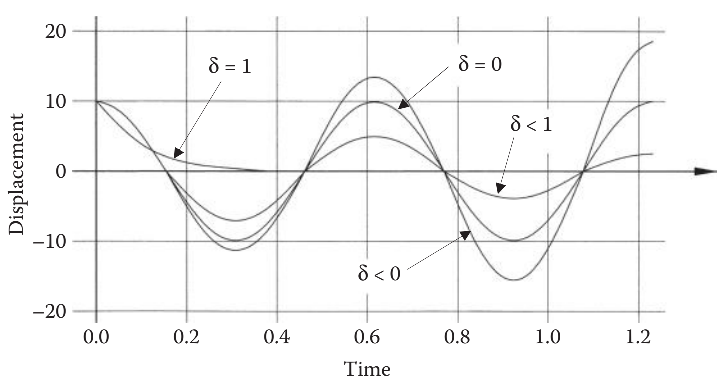


FIGURE 6.19 Comparison of various damping ratios.

An excellent example is that of a child sitting on a swing. If the child begins to swing but does nothing, the swing will gradually come to a halt. If someone gives the swing a small push at the start of each swing, energy is being added to the system and the swing (oscillations) begins to increase, that is, gets higher and higher. This phenomenon is called excitation.

Many engineering structures are prone to vibration when being excited at or near their natural frequency. If the excitation is close to the natural frequency, then the oscillations may become out of control. An example is the wind blowing around cooling towers, chimney stacks and suspended cables. The phenomenon is known as vortex shedding and is a major problem for the designers of these structures as the oscillations may increase and lead to a catastrophic failure.

At a lower level, if an automotive is travelling over a corrugated surface such as Belgian pave, the disturbance may be close to the natural frequency of the suspension causing the vehicle to possibly bounce out of control. Most vehicle-testing laboratories will usually have a section of pave as part of the suspension test.

6.4.1 Forced Vibrations

Two types of forced vibrations will be considered:

1. When a mass has a disturbing force acting on it.
2. When the spring support is disturbed harmonically.

6.4.1.1 Disturbing Force Acting on Mass

Figure 6.20 shows a system where a mass sitting on a spring and fitted with a damper has a fixed support. Located on the mass is a small rotating machine that is out of balance. This is equivalent to a small mass 'm' that is rotating at a radius 'r' and is producing the out-of-balance force. The magnitude of the force is $F_o = mr\omega^2$. The main mass is restrained within guides and is restricted to be only able to move vertically (one degree of freedom). At the position shown with the small mass rotated through an angle of 'θ', the component of F_o acting vertically is $F' = F_o \sin\theta$.

Any force applied to the main mass must overcome the inertia, damping force and spring force.

The applied force is

$$F = F_i + F_d + F_s \tag{6.62}$$

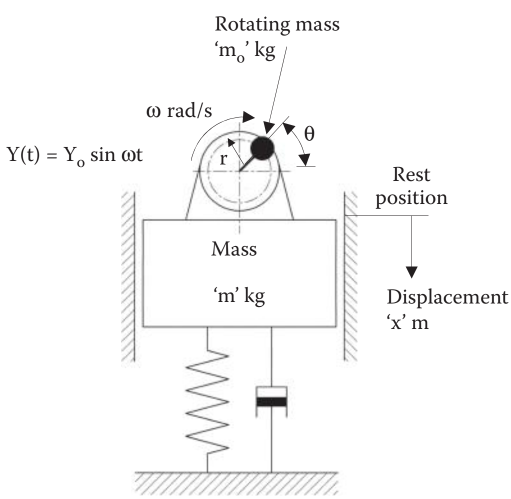


FIGURE 6.20 Mass subject to rotor excitation.

$$F = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx \tag{6.63}$$

In this case, as the mass is restricted to move vertically, the only force applied to the mass is the vertical component of the centrifugal force.

$$F_o \sin(\omega t) = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx \tag{6.64}$$

6.4.1.2 Phasor Representation

It is assumed that the mass oscillates vertically with a sinusoidal oscillation 'A'. Assume that the oscillation is going to start when it passes through the rest position. The displacement is given by $x = A\sin \omega t$ where A is the amplitude.

The velocity is $v = (dx/dt) = A\omega\cos \omega t$ where $A\omega$ is the amplitude.

The acceleration will be $a = (dv/dt) = -A\omega^2\sin \omega t$ where $A\omega^2$ is the amplitude.

The displacement 'x', velocity 'v' and acceleration 'a' when plotted against time 't' will result in Figure 6.21. Each graph may be generated by a vector rotating at ω rad/s and with a length equivalent to the amplitude. Such vectors are referred to as phasors. At a given point in time, the tip of each vector is projected across to the appropriate point as indicated.

It is clearly seen that in order to produce the results, the velocity vector is 90° in front of the displacement vector and the acceleration vector is 90° in front of the velocity vector.

Considering Figure 6.20, it is obvious that the spring force is directly proportional to the displacement 'x', so therefore it must be in phase with 'x'. The damping force is then directly proportional to the velocity 'v' and is in phase with 'v'; and finally, the inertial force is directly proportional to the acceleration 'a'; hence, it follows that the inertia is in phase with 'a'.

All three forces can then be represented by phasors rotating at an angular velocity ω rad/s.

The spring force is in phase with the movement; therefore, the displacement vector can be drawn horizontally and the velocity vector and acceleration vector can be drawn 90° and 180° ahead, respectively as shown in Figure 6.22.

The sum of these three forces can be represented by a force F_o and by adding these three forces a typical vector diagram will result as shown in Figure 6.23.

From the diagram, it is seen that the applied force F_o is at an angle ϕ to the horizontal axis; therefore, it must be displaced by a phase angle 'ϕ' relative to 'x'.

$$F_o^2 = (kA - MA\omega^2)^2 + (cA\omega)^2 \tag{6.65}$$

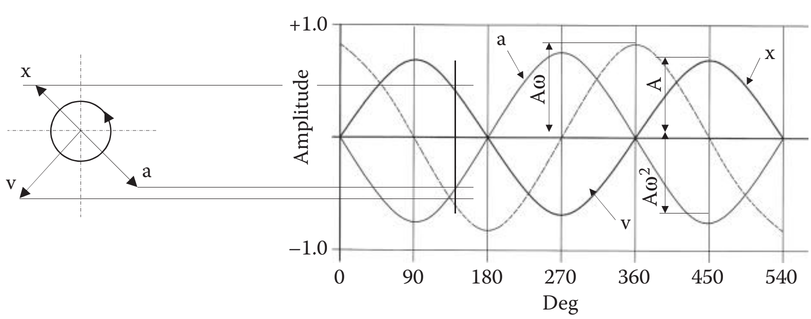


FIGURE 6.21 Phasor representation.

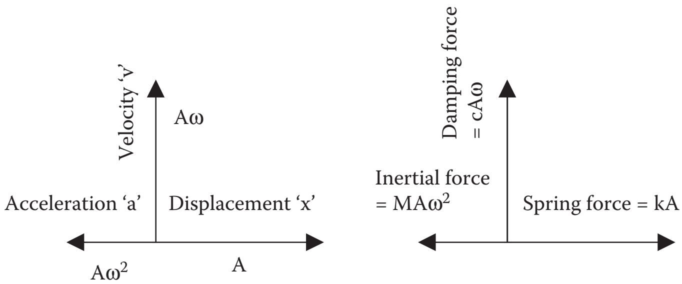


FIGURE 6.22 Representation of vectors.

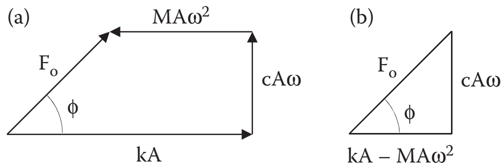


FIGURE 6.23 (a) and (b): Addition of vectors.

$$F_o^2 = A^2(k - M\omega^2)^2 + A^2(c\omega)^2 \quad (6.66)$$

$$F_o^2 = A^2\left[(k - M\omega^2)^2 + (c\omega)^2\right] \quad (6.67)$$

Dividing through by M^2

$$\frac{F_o^2}{M^2} = A^2\left[(k - \omega^2)^2 + \left(\frac{c\omega}{M}\right)^2\right] \quad (6.68)$$

From Equations 6.50 and 6.52, it was shown that $\omega_n^2 = k/m$ and $c/M = 2\delta\,\omega_n$.

$$A^2 = \left(\frac{F_o}{M}\right)^2 \left\{\frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega\omega_n)^2}\right\} \quad (6.69)$$

From the triangle, the phase angle 'ϕ' is obtained.

$$\tan\phi = \frac{2\delta\omega\omega_n}{\omega_n^2 - \omega^2} \quad (6.70)$$

Plotting displacement (x) and the applied force (F_o) against time gives a graph similar to Figure 6.24.

6.5 NATURAL FREQUENCY OF BEAMS AND SHAFTS

This section discusses the natural frequency of transverse vibrations.

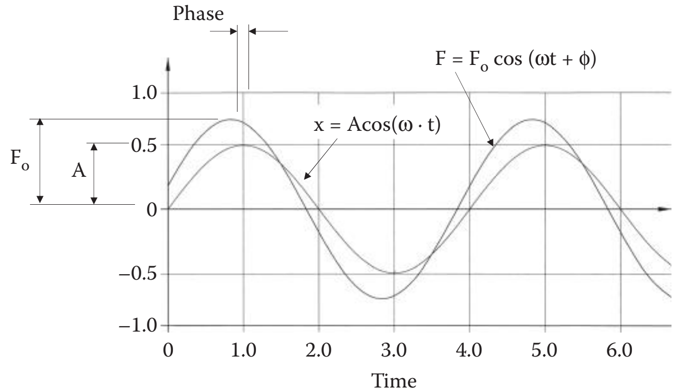


FIGURE 6.24 Frequency versus time.

6.5.1 Degrees of Freedom

In Section 6.3, vibration in one degree of freedom was discussed. In this section, this discussion will be extended to three degrees of freedom.

There are three translation axes as shown in Figure 6.25; they are in the 'x', 'y' and 'z' directions in which an object can move unrestrained.

There are three other degrees of freedom which cover pitch, roll and yaw. These are superimposed on the translation axes as shown in Figure 6.26.

In the study of free vibrations, only one degree of freedom will be considered.

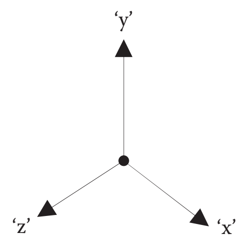


FIGURE 6.25 Three degrees of freedom axes.

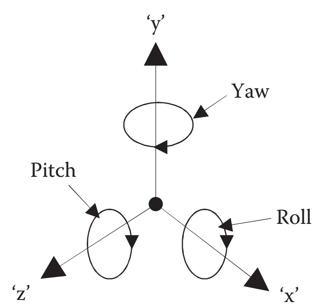


FIGURE 6.26 Six degrees of freedom.

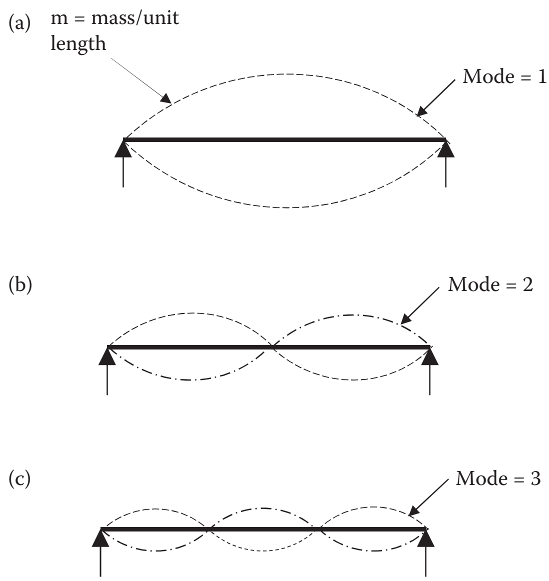


FIGURE 6.27 Transverse harmonics. (a) First harmonic, (b) second harmonic and (c) third harmonic.

6.5.2 Beams Subject to Transverse Vibrations

A beam or cantilever when subject to transverse vibrations will exhibit various modal behaviour characteristics. These behaviours will be dependent upon the natural frequency of the beam or cantilever.

6.5.3 Simply Supported Beam Subject to Transverse Vibration

Consider a beam as shown in Figure 6.27a which is simply supported at each end and subjected to a transverse vibration. This is referred to as a mode-1 vibration.

If the frequency of the vibration is increased, the second harmonic is reached (see Figure 6.27b). And if the frequency is increased even further, a third harmonic will be reached resulting in Figure 6.27c.

6.5.4 Torsional Frequency of a Cantilevered Shaft Carrying a Mass at the Free End (Figure 6.28)

Assume the beam has a negligible mass.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{J_m}}\ \text{Hz} \tag{6.71}$$

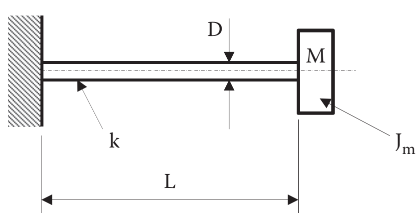


FIGURE 6.28 Torsional frequency of a cantilever shaft.

where

k = torsional stiffness of shaft $= \frac{G \cdot J}{L}$

J = polar moment of inertia $= \frac{\pi}{32} D^4$

J_m = polar moment of inertia of mass = Mk^2

6.5.5 Torsional Frequency of a Shaft Carrying Two Masses (Figure 6.29)

Assume the beam has a negligible mass.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_t(J_1 + J_2)}{(J_1 \cdot J_2)}}\ \text{Hz} \tag{6.72}$$

Torsional stiffness of shaft $= \frac{G \cdot J}{L}$(Nm/rad)
G = shear modulus (N/m^2)
θ = twist in the shaft (rad)
Polar moment of inertia of masses = $M_1k^2 + M_2k^2$
Position of node $= L_1 = L/(1 + (J_{M_1}/J_{M_2}))$
Ratio of the twist in the shaft

$$\frac{\theta_1}{\theta_2} = \frac{J_2}{J_1} \tag{6.73}$$

Figure 6.30 shows the position of the node.

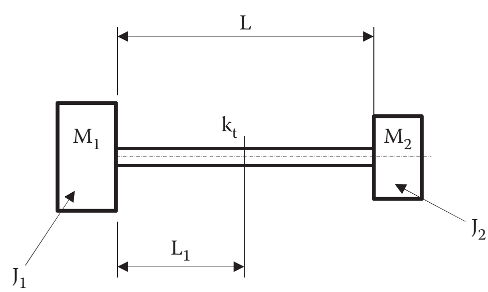


FIGURE 6.29 Torsional frequency of shaft carrying two masses.

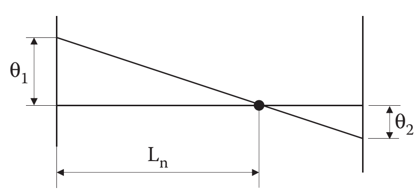


FIGURE 6.30 Natural frequency of a two mass system and node position.

6.5.6 Torsionally Equivalent Shafts

In the previous two examples, a uniform diameter shaft was considered. In actual practice, shafts with variable diameters and lengths are more likely to be used.

In such cases, a shaft may theoretically be replaced by an equivalent shaft that has a uniform diameter.

Figure 6.31a depicts a shaft having three steps with diameters d_1, d_2 and d_3 with lengths L_1, L_2 and L_3, respectively. Consider that this shaft is replaced by an equivalent shaft having a uniform diameter 'd_e' and length 'L_e' as shown in Figure 6.31b. The equivalent shaft has to have the same angle of twist as the original shaft when opposing torques 'T' are applied from both ends of the shaft.

Let

d_1, d_2 and d_3 = diameters for the corresponding lengths 'L'.
θ_1, θ_2 and θ_3 = angle of twist for lengths L_1, L_2 and L_3, respectively.
θ_T = total angle of twist.
J_1, J_2 and J_3 = polar moments of inertia for the shafts having diameters d_1, d_2 and d_3, respectively.

As the total angle of twist of the equivalent shaft is equal to the sum of the individual angles of twists of the different lengths, therefore

$$\theta_1 = \frac{T \cdot L_1}{G \cdot J_1}, \quad \theta_2 = \frac{T \cdot L_2}{G \cdot J_2}, \quad \theta_3 = \frac{T \cdot L_3}{G \cdot J_3} \tag{6.74}$$

Hence

$$\frac{T \cdot L_e}{G \cdot J_e} = \frac{T \cdot L_1}{G \cdot J_1} + \frac{T \cdot L_2}{G \cdot J_2} + \frac{T \cdot L_3}{G \cdot J_3} \tag{6.75}$$

Extracting T/G which is common throughout:

$$\frac{L_e}{J_e} = \frac{L_1}{J_1} + \frac{L_2}{J_2} + \frac{L_3}{J_3} \tag{6.76}$$

Now

$$J_e = \frac{\pi \cdot d_e^4}{32}, \quad J_1 = \frac{\pi \cdot d_1^4}{32}, \quad J_2 = \frac{\pi \cdot d_2^4}{32} \quad \text{and} \quad J_3 = \frac{\pi \cdot d_3^4}{32} \tag{6.77}$$

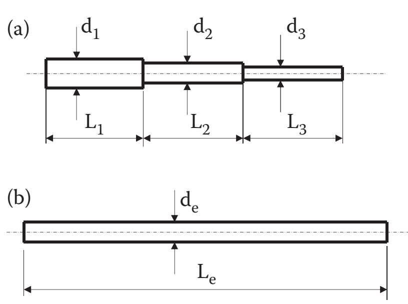


FIGURE 6.31 (a) and (b): Three-stepped equivalent shaft.

Replacing 'J' and multiplying throughout with d_1^4 where the diameter of the equivalent shaft diameter 'd' is taken as d_1, the length of the equivalent shaft is

$$L_e = L_1 + L_2 \frac{d_1^4}{d_2^4} + L_3 \frac{d_1^4}{d_3^4} \tag{6.78}$$

Whenever confronted with a stepped shaft in torsional vibration problems, always convert it into an equivalent shaft of uniform diameter and then determine the torsional frequency as discussed for the two rotor example.

EXAMPLE 6.7

Consider a steel shaft as illustrated in Figure 6.32. The steel shaft is 1500 mm long. It is stepped at 95 mm diameter for a length of 600 mm, followed by a diameter of 60 mm for a length of 500 mm and finally a diameter of 50 mm for the remaining length of 400 mm. The shaft is fitted with two flywheels at each end. The first flywheel has a mass of 900 kg and a radius of gyration of 850 mm. This flywheel is located on the 95 mm diameter portion of the shaft. The second flywheel has a mass of 700 kg with a radius of gyration of 550 mm and is located at the 50 mm diameter portion of the shaft.

Calculate the position of the node together with the natural frequency of the free torsional vibration of the shaft. The modulus of rigidity (G) of the shaft material may be taken as 7.2 GPa.

Solution

Given that

L = 1500 mm	
L_1 = 600 mm	d_1 = 95 mm
L_2 = 500 mm	d_2 = 60 mm
L_3 = 400 mm	d_3 = 50 mm
M_a = 900 kg	k_a = 850 mm
M_b = 700 kg	k_b = 550 mm
G = 7.2 GPa	

Determine the length of the equivalent shaft assuming its diameter as d_1 = 95 mm.

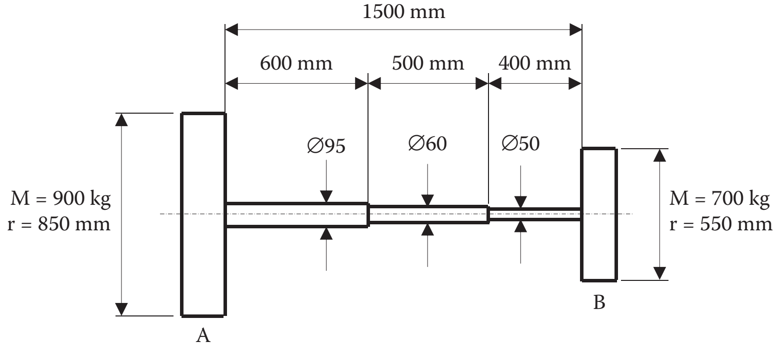


FIGURE 6.32 Stepped shaft carrying two flywheels.

From Equation 6.78, the length of the equivalent shaft is calculated as

$$L_e = L_1 + L_2\frac{d_1^{\,4}}{d_2^{\,4}} + L_3\frac{d_1^{\,4}}{d_3^{\,4}}$$

$$L_e = 600\text{ mm} + 500\text{ mm} \times \frac{95\text{ mm}^4}{60\text{ mm}^4} + 400\text{ mm} \times \frac{95\text{ mm}^4}{50\text{ mm}^4}$$

$$L_e = 8955\text{ mm (8.955 m)}$$

Location of the Node

Assume the position of the node of the equivalent shaft lies at 'N' as shown in Figure 6.33.
Let

L_A = distance of the node from the flywheel 'A'
L_B = distance of the node from the flywheel 'B'

Mass moment of inertia of flywheel 'A':

$$J_A = M_a \times k_a^{\,2}$$

$$J_A = 900\text{ kg} \times (850\text{ mm})^2$$

$$J_A = 650.25\text{ kg}\cdot\text{m}^2$$

Mass moment of inertia of flywheel 'B':

$$J_B = M_b \times k_b^{\,2}$$

$$J_B = 700\text{ kg} \times (550\text{ mm})^2$$

$$J_B = 211.75\text{ kg}\cdot\text{m}^2$$

Hence,

$$L_A \times J_A = L_B \times J_B$$

$$L_A = \frac{L_B \times J_B}{J_A}$$

$$L_A = \frac{L_B \times 211.75\text{ kg}\cdot\text{m}^2}{650\text{ kg}\cdot\text{m}^2}$$

$$L_A = 0.326\,L_B$$

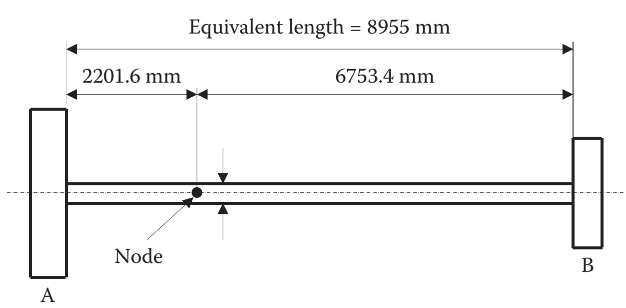


FIGURE 6.33 Mode position for Example 6.7.

Now

$$L_A + L_B = L = 8955 \text{ mm}$$

$$L_A = 0.326\ L_B + L_B$$

$$L_A = 1.326\ L_B$$

$$L_B = \frac{8955 \text{ mm}}{1.326}$$

$$\therefore \quad L_B = 6753.4 \text{ mm}$$

$$L_A = 8955 \text{ mm} - 6753.4 \text{ mm}$$

$$L_A = 2201.6 \text{ mm}$$

Hence, the node lies at 2201.6 mm from flywheel 'A' and 6753.4 mm from flywheel 'B'.

Natural Frequency of Free Torsional Vibrations

The polar moment of inertia of the equivalent shaft:

$$J_e = \frac{\pi \cdot d_1^4}{32}$$

$$J_e = \frac{\pi \times (0.095 \text{ m})^4}{32}$$

$$J_e = 8.0 \times 10^{-6} \text{ m}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{G \cdot J_e}{L_A \cdot J_B}}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{72 \times 10^9 \text{N/m}^2 \times 8.0 \times 10^{-6} \text{ m}^4}{2.2 \text{ m} \times 650 \text{ kg} \cdot \text{m}^2}}$$

$$f_n = 3.194 \text{ Hz}$$

6.5.7 Torsional Frequency of a Geared Shaft Carrying Two Masses (Figure 6.34)

Note: The shafts are assumed to be light.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{G_R^2 k_2 + k_1}\left(\frac{J_1 + G_R^2 J_2}{J_1 \cdot J_2}\right)} \text{Hz} \quad \text{(Neglecting the inertias of the gears)} \tag{6.79}$$

where

$$G_R = \text{Gear ratio} = \frac{\text{Speed of shaft 2}}{\text{Speed of shaft 1}} \tag{6.80}$$

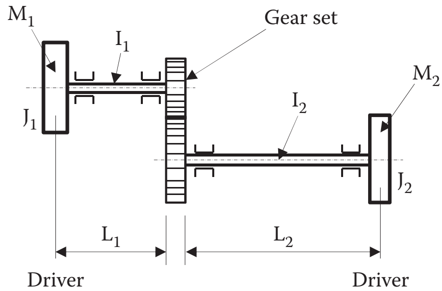


FIGURE 6.34 Torsional frequency of a geared shaft with two masses.

$$k_1 = \frac{G_{R1} I_1}{L_1} \quad k_2 = \frac{G_{R2} I_2}{L_2} \tag{6.81}$$

Note: Torque and inertia are referred to the driven shaft.

Torque on shaft 2 = torque on shaft 1 × (gear ratio) (G_R).

Inertia of shaft 2 = inertia of shaft 1 × (gear ratio)^2(G_R)^2.

EXAMPLE 6.8

A steel shaft 100.0 mm in diameter and 1000 mm long carries a flywheel at its end measuring 1000 mm in diameter and weighing 1000 kg.

Calculate the torsional frequency of this single rotor system (see Figure 6.35).

Solution

Given that the diameter of the shaft = 100 mm, length of the shaft = 1000 mm, modulus of rigidity (G) = 72 × 10^9 Pa and density (ρ) = 7.8 kg/m^3.

Polar moment of inertia of the shaft:

$$J_{shaft} = \frac{\pi \times (\text{dia' shaft})^4}{32}$$

$$J_{shaft} = \frac{\pi \times 100\ \text{mm}^4}{32} = 9.817 \times 10^{-6}\ \text{m}^4$$

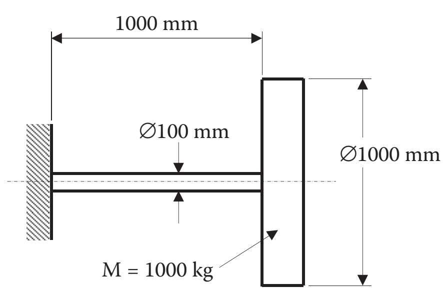


FIGURE 6.35 Torsional frequency of a plane shaft with one flywheel.

Stiffness of the shaft:

$$S = \frac{J \cdot G}{L}$$

$$S = \frac{9.817 \times 10^{-6}\ m^4 \times 72 \times 10^9\ Pa}{1000\ mm} = 7.069 \times 10^5\ N \cdot m$$

Polar Moment of Inertia of the Flywheel

Mass moment of inertia:

Mass = 1000 kg
Radius of gyration = 500 mm

$$J_m = M \cdot k^2$$

$$J_m = 1000\ kg \times 500^2\ mm = 250\ kg \cdot m^2$$

$$f_n = \frac{1}{2\pi} \cdot \sqrt{\frac{S}{J_m}}$$

$$f_n = \frac{1}{2\pi} \cdot \sqrt{\frac{7.069 \times 10^5\ Nm}{250\ kg \cdot m^2}}$$

$$f_n = 8.463\ Hz$$

EXAMPLE 6.9

A steel shaft 100.0 mm in diameter and 1000 mm long carries two identical flywheels at each end measuring 1000 mm in diameter and weighing 1000 kg each.

Calculate the torsional frequency of this two rotor system and determine the node position. (Refer to Figure 6.36).

Solution

Given that the diameter of the shaft = 100 mm, length of the shaft = 1000 mm, modulus of rigidity (G) = 72×10^9 Pa and density (ρ) = 7.8 kg/m^3.

Polar Moment of Inertia of Shaft

$$J_{shaft} = \frac{\pi \times (\text{dia' shaft})^4}{32}$$

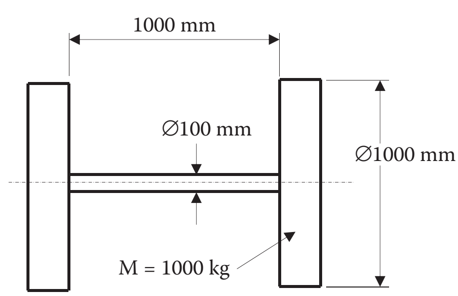


FIGURE 6.36 Torsional frequency of a plane shaft with two flywheels.

$$J_{shaft} = \frac{\pi \times 100 \text{ mm}^4}{32} = 9.817 \times 10^{-6} \text{ m}^4$$

Stiffness of the shaft

$$S = \frac{J \cdot G}{L}$$

$$S = \frac{9.817 \times 10^{-6} \text{ m}^4 \times 72 \times 10^9 \text{ Pa}}{1000 \text{ mm}} = 7.069 \times 10^5 \text{ N} \cdot \text{m}$$

Polar Moment of Inertia of Flywheels

Mass moment of inertia:

Mass = 1000 kg
Radius of gyration = 500 mm

$$J = M \cdot k^2$$
$$J = 1000 \text{ kg} \times 500^2 \text{ mm} = 250 \text{ kg} \cdot \text{m}^2$$

Torsional Frequency

$$f_n = \frac{1}{2\pi}\sqrt{\frac{S \cdot (2J)}{J^2}}$$
$$f_n = \frac{1}{2p}\sqrt{\frac{7.069 \times 10^5 \times (2 \times 250 \text{ kg} \cdot \text{m}^2)}{(250 \text{ kg} \cdot \text{m}^2)^2}} = 11.968 \text{ Hz}$$

Position of Node

$$L_n = \frac{1000 \text{ mm}}{(1 + (250 \text{ kg} \cdot \text{m}^2 / 250 \text{ kg} \cdot \text{m}^2))}$$
$$L_n = 500 \text{ mm (from either end)}$$

EXAMPLE 6.10

In this exercise, a shaft connects a motor to a gear set and a further shaft then connects to a pump.

The pump speed is one-third of that of the motor. The diameter of the shaft from the motor to the pinion of the gear set is 60 mm and the length of the shaft is 300 mm. The moment of inertia of the motor is 400 kg · m^2. The pump shaft is 100 mm in diameter and is 600 mm long. The moment of inertia of the pump is 1500 kg · m^2. Neglecting the inertia of the shafts and gears, determine the frequency of torsional vibration of the system. The modulus of rigidity of the shaft material is 72 GPa.

Solution

Given that

$G_r = N_a/N_b = 3{:}1$ (G_r = gear ratio), $d_1 = 60$ mm, $L_1 = 300$ mm, $d_2 = 100$ mm, $L_2 = 600$ mm, $J_a = 400$ kg · m^2, $J_b = 1500$ kg · m^2 and $G = 72 \times 10^9$ N/m^2.

The first action is to establish the moment of inertia of the equivalent rotor 'B' and the additional length of the equivalent shaft assuming its diameter is $d_1 = 60$ mm.

The Moment of Inertia of the Equivalent Rotor 'B'

$$J_b = \frac{J_b}{Gr^2}$$

$$J_b = \frac{1500\ kg \cdot m^2}{3^2}$$

$$J_b = 166.667\ kg \cdot m^2$$

The Additional Length of the Equivalent Shaft

$$L_3 = G_r^{\,2} \cdot L_2 \cdot \left(\frac{d_1}{d_2}\right)^4$$

$$L_3 = 3^2 \times 600\ mm \times \left(\frac{60\ mm}{100\ mm}\right)^4$$

$$L_3 = 700\ mm$$

The Total Length of the Equivalent Shaft

$$L_e = L_1 + L_3$$

$$L_e = 1000\ mm$$

Let the node of the equivalent system lie at position 'N' as shown in Figure 6.37a and b.

$$L_a \times J_a = L_b \times J_b$$

$$L_a = \frac{L_b \cdot J_b}{J_a}$$

$$L_a = \frac{700\ mm \times 166.67\ kg \cdot m^2}{400\ kg \cdot m^2}$$

$$L_a = 291.67\ mm$$

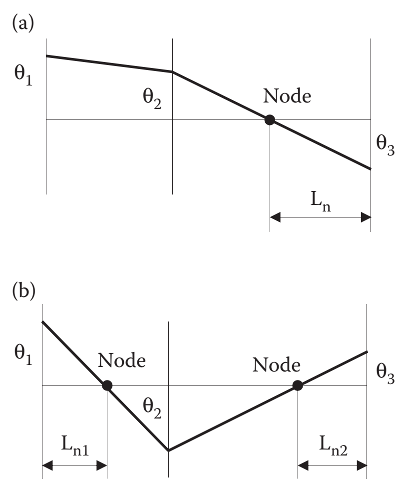


FIGURE 6.37 (a) and (b): First and second modes for Example 6.10.

The Polar Moment of Inertia of the Equivalent Shaft

$$J_e = \frac{\pi}{32} d_1^4$$

$$J_e = \frac{\pi}{32} \times (60\ \text{mm})^4$$

$$J_e = 1.27 \times 10^6\ \text{mm}^4$$

Torsional Frequency of the Equivalent Shaft

$$L_e = L_1 + L_2 \frac{d_1^4}{d_2^4} + L_3 \frac{d_1^4}{d_3^4}$$

$$f_n = \frac{1}{2\pi} \sqrt{\frac{G \cdot J_e}{L_a \cdot J_a}}$$

$$f_n = \frac{1}{2\pi} \sqrt{\frac{72 \times 10^9 P_a \times 1.272 \times 10^6\ \text{mm}^4}{291.67\ \text{mm} \times 400\ \text{kg} \cdot \text{m}^2}}$$

$$f_n = 4.46\ \text{Hz}$$

6.5.8 Torsional Frequency of a Shaft Carrying Three Masses (Figure 6.38)

Note: The shaft is assumed to be light.

First mode (low frequency):

$$F_{n1} = \frac{1}{2\pi} \sqrt{\frac{1}{2} \left[(c_1 + c_2) - \sqrt{(c_1 - c_2)^2 + 4 \frac{k_1 \cdot k_2}{J_2^2}} \right]} \tag{6.82}$$

where

$$k_1 = \frac{G_1 \cdot I_1}{L_1} \quad \text{and} \quad k_2 = \frac{G_2 \cdot I_2}{L_2}$$

Ratio of twists in the shaft:

$$\frac{\theta_2}{\theta_3} = \frac{k_2 - J_3\omega_1^2}{k_2}; \quad \frac{\theta_3}{\theta_1} = \frac{k_2(k_1 - J_1\omega_1^2)}{k_1(k_2 - J_3\omega_1^2)}; \quad \frac{\theta_2}{\theta_1} = \frac{k_1 - J_1\omega_1^2}{k_1} \tag{6.83}$$

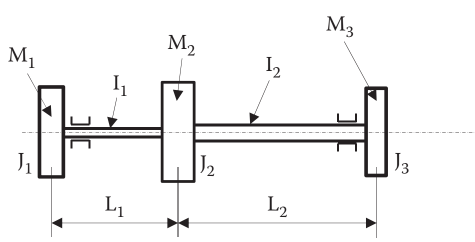


FIGURE 6.38 Torsional frequency of a shaft carrying three masses.

Position of node:

$$L_n = \frac{L_2}{(\theta_2/\theta_1)+1} \tag{6.84}$$

$$F_{n2} = \frac{1}{2\pi}\sqrt{\frac{1}{2}\left[(c_1+c_2)+\sqrt{(c_1-c_2)^2+4\frac{k_1\cdot k_2}{J_2^{\,2}}}\right]} \tag{6.85}$$

Ratio of twists in the shaft:

$$\frac{\theta_2}{\theta_3} = \frac{k_2 - J_3\omega_2^2}{k_2}; \quad \frac{\theta_3}{\theta_1} = \frac{k_2(k_1 - J_1\omega_2^2)}{k_1(k_2 - J_3\omega_2^2)}; \quad \frac{\theta_2}{\theta_1} = \frac{k_1 - J_1\omega_2^2}{k_1} \tag{6.86}$$

$$k_1 = \frac{G_1\cdot I_1}{L_1}; \quad k_2 = \frac{G_2\cdot I_2}{L_2} \tag{6.87}$$

$$c_1 = k_1\left(\frac{1}{J_1}+\frac{1}{J_2}\right); \quad c_2 = k_2\left(\frac{1}{J_2}+\frac{1}{J_3}\right) \tag{6.88}$$

Position of nodes:

$$L_{n1} = \frac{L_1}{(\theta_2/\theta_1)+1}; \quad L_{n2} = \frac{L_2}{(\theta_2/\theta_3)+1} \tag{6.89}$$

6.6 FORCED VIBRATIONS

6.6.1 Overview

In Section 6.3, which covers damped vibrations, it was demonstrated that a free vibration will diminish over time as the energy contained in the vibrating system is dissipated by the damping. The displacement in a damped oscillation is given as

$$x = Ce^{-\delta\omega_n t}\cos(\omega t) \tag{6.90}$$

where

δ = damping ratio and ω_n = natural angular frequency

1. When $\delta > 1$, the system is overdamped.
2. When $\delta = 1$, the system is critically damped.
3. When $\delta < 1$, the damped oscillation will gradually diminish in time.
4. When $\delta = 0$, the system is undamped and steady oscillation will occur.

Figure 6.19 compares the effects of various damping factors on oscillations.

When the damping ratio (δ) is less than zero, that is, negative, instead of energy being taken out of the system, there is an external force inputting energy into the system. As the energy is added, the amplitude increases. When this energy is added, such oscillations are said to be ‘forced’. A classic example of a forced oscillation is that of a child on a swing. When the child is given an initial push to start the swing, the oscillations will gradually decrease due to air resistance on the child's body together with friction at the swing supports and will eventually come to a halt. If the swing is given a slight push at the start of each swing, energy is being added to the system and the swing will go higher and higher. This phenomenon is known as excitation.

Two further examples of forced vibrations are when an automobile is being driven over Belgian pave. In some cases, the vehicle's suspension is not able to cope with the rough ride and there is excessive vibration and the suspension oscillations will increase uncontrollably. The second example happens when a vehicle's wheels are out of balance and that creates excessive vibration of the wheel.

There are many other examples in engineering structures that vibrate at or near the natural frequency of the structure such as a pump and motor set on a suspended floor, suspension bridges and chimney columns subject to a wind blowing around them causing vortex shedding.

There are three types of forcing mechanisms that will be considered and these will be applied to a spring–mass system as shown in Figure 6.39a through c. The masses are constrained in the horizontal plane and will be able to move only in the vertical plane; hence, they are subjected to one degree-of-freedom.

6.6.2 External Forcing

Figure 6.39a models the behaviour of a system that has a time-varying force acting on it such as a structure being subject to wind loading.

Base excitation: This system model represents the behaviour of a vibration isolation system where the base of the spring is given a prescribed motion, causing the mass to vibrate. The arrangement is shown in Figure 6.39b. Examples include a vehicle suspension system or the earthquake response of a structure.

Rotor excitation: Consider a motor attached to a large mass (M) and fitted with an out-of-balance mass (m_o) that rotates at a constant speed at radius 'r' as depicted in Figure 6.39c. This causes the main mass 'M' to oscillate. In this case, the disturbing force will be harmonic, that is, sinusoidal.

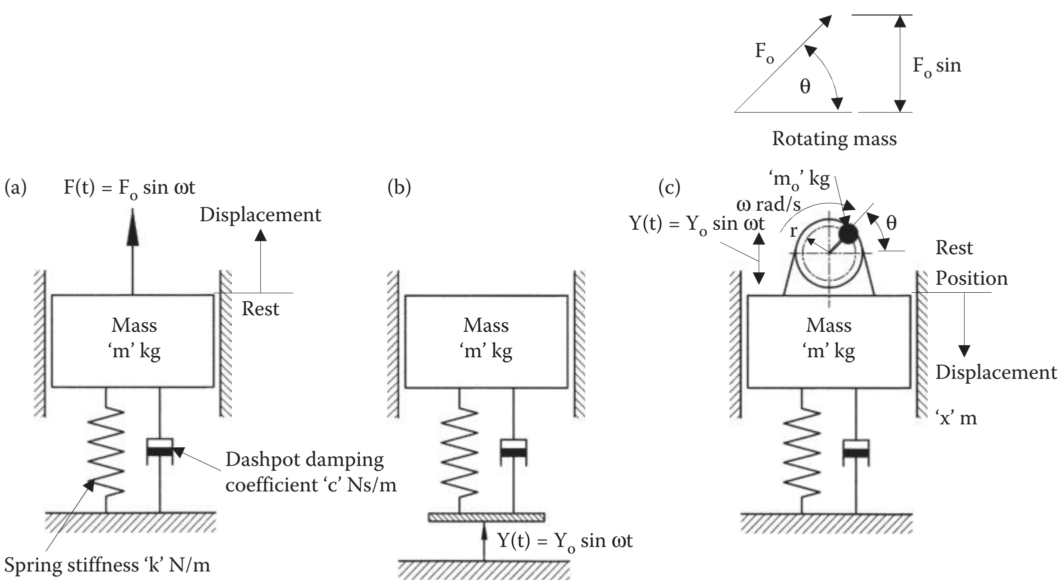


FIGURE 6.39 External forcing examples. (a) External forcing, (b) base excitation and (c) rotor excitation.

Any force that is applied to the mass to make it move has to overcome the inertia (F_i) of the mass, spring force (F_s) from the supporting structure and damping (F_d). The applied force (F_a) will be

$$F_a = F_i + F_s + F_d \tag{6.91}$$

$$F = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx \tag{6.92}$$

In the cases shown, the mass can only move vertically; therefore, the only force applied to it will be the vertical component of the centrifugal force:

$$F_o \sin(\omega t) = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx \tag{6.93}$$

When the mass is subjected to a harmonic disturbing force, it will oscillate vertically with a sinusoidal motion having amplitude 'A'. Consider that the timing of the oscillation begins when the oscillation passes through the rest position. The displacement is given by

$$x = A \sin \omega t \tag{6.94}$$

where A is the amplitude.

The velocity will be

$$v = \frac{dx}{dt} = A\omega \cos \omega t \tag{6.95}$$

where $A\omega$ is the amplitude.

The acceleration will be

$$a = \frac{dv}{dt} = -A\omega^2 \sin \omega t \tag{6.96}$$

where $A\omega^2$ is the amplitude.

In Figure 6.21, the displacement 'x', velocity 'v' and acceleration 'a' are plotted against time. Each graph is generated by a vector rotating at ω rad/s with a length equal to the amplitude.

Such vectors are called phasors. At a given point in time, the tip of each vector is projected across to the appropriate point in the graph as shown in the figure.

For this result, the velocity vector has to be 90° in front of the displacement and the acceleration needs to be 90° in advance of the velocity.

The spring force will be directly proportional to the physical displacement 'x'; therefore, it has to be in phase with the displacement 'x'. The damping force is directly proportional to the velocity 'v' and therefore has to be in phase with the velocity 'v'. The inertial force is directly proportional to the acceleration 'a' and therefore will be in phase with the acceleration 'a'.

The three forces can be represented by phasors rotating at an angular velocity ω rad/s, choosing a time when the displacement is horizontal as shown in Figure 6.22a.

The spring force being in phase with the displacement is drawn horizontally. The other vectors will be 90° and 180° ahead, respectively, as depicted in Figure 6.22b.

The sum of these three vectors is F_o; adding these together, a typical vector diagram as shown in Figure 6.23 will be obtained.

It will be seen from the figure that the applied force F_o is at an angle 'θ' to the horizontal; hence, it must be displaced by the phase angle 'θ' relative to 'x'.

From Pythagoras's theorem

$$F_o^2 = A^2(k - M \cdot \omega^2)^2 + A^2(c \cdot \omega)^2 \quad \text{(Extracting } A^2\text{)} \tag{6.97}$$

$$F_o^2 = A^2\left[(k - M \cdot \omega^2)^2 + (c \cdot \omega)^2\right] \quad \text{(Simplifying)} \tag{6.98}$$

$$\frac{F_o^2}{M^2} = A^2\left[\left(\frac{k}{M} - \omega^2\right)^2 + \left(\frac{c \cdot \omega}{M}\right)^2\right] \quad \text{(Dividing each term by } M^2\text{)} \tag{6.99}$$

Now

$$w_n^2 = \frac{k}{M} \quad \text{and} \quad \frac{c}{M} = 2\delta\omega_n$$

$$A^2 = \left(\frac{F_o}{M}\right)^2\left[\left(\frac{1}{(\omega_n^2 - \omega^2)^2 + \left(2\delta\omega \cdot \omega n\right)^2}\right)\right] \tag{6.100}$$

From Figure 6.24, it is possible to deduce the phase angle

$$\text{Tan } \phi = \frac{2\delta \cdot \omega \cdot \omega_n}{\omega_n^2 - \omega^2} \tag{6.101}$$

6.6.3 Frequency Response Diagrams

Consider the spring–mass–damper model of Figure 6.39a with a harmonic force externally applied.

This type of force could, for example, be generated by a rotating imbalance in, say, a motor or a wheel on a vehicle. If the speed of rotation (ω) is gradually increased from zero and taking a value of $\omega_n = 10$, plotting 'θ' against 'ω' for various values of 'δ', a graph like Figure 6.40 will result.

The graph shows that the phase angle 'θ' starts at zero and reaches 90° when $\omega = \omega_n$. As the speed of rotation is increased, the phase angle approaches 180°.

Plotting the amplitude 'A' against frequency 'ω' for various values of 'δ' will result in the graph shown in Figure 6.41.

Equation 6.99 is analysed at three frequencies.

When

1. $\omega = 0$
2. $\omega = \omega_n$
3. $\omega > \omega_n$

$$A^2 = \left(\frac{F_o}{M}\right)^2\left[\left(\frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega \cdot \omega n)^2}\right)\right] \tag{6.102}$$

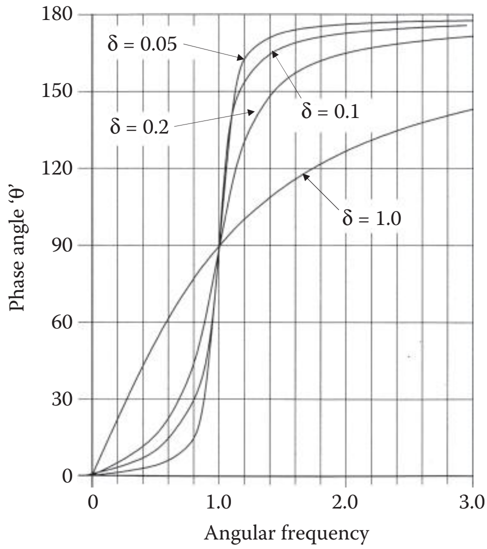


FIGURE 6.40 Phase angle with respect to damping.

1. $\omega = 0$: The equation reduces to

$$A = \left(\frac{F_o}{M}\right)\left\{\frac{1}{\omega_n^2}\right\} \tag{6.103}$$

   This will have a finite value (1 in the Figure 6.41) and will be the same starting point for all values of 'δ'.

2. $\omega = \omega_n$: At this point, the amplitude will become

$$A = \left(\frac{F_o}{M}\right)\left\{\frac{1}{(2\delta\omega^2)}\right\}$$

   The value of the amplitude will depend upon the value of 'δ'. The smaller the value of damping (δ), the larger the peak value of 'A'. If the damping factor 'δ' is reduced to zero, then theoretically 'A' $\rightarrow \infty$.

3. $\omega > \omega_n$: As the frequency increases past 'ω_n', the amplitude will begin to diminish down to zero for all values of 'δ'.

   It can be concluded that when an out-of-balance machine rotates at a speed very much greater than 'ω_n', there will be very little disturbance to the system, but when the speed of rotation is reduced and approaches ω_n, the amplitude will increase and could possibly become very large. In the past, it has led to catastrophic failures, so care has to be exercised. As an aside, very large armatures as used in electrical power stations are designed to have a low resonant frequency and when starting up have to pass through all the harmonic frequencies. Before it settles down at its operating speed, the armature will remain at this speed for many months to minimise any possible damage when slowing down. It should be noted that the frequency at which the amplitude peaks is known as the resonant frequency, which is not quite the same as the natural frequency of the system.

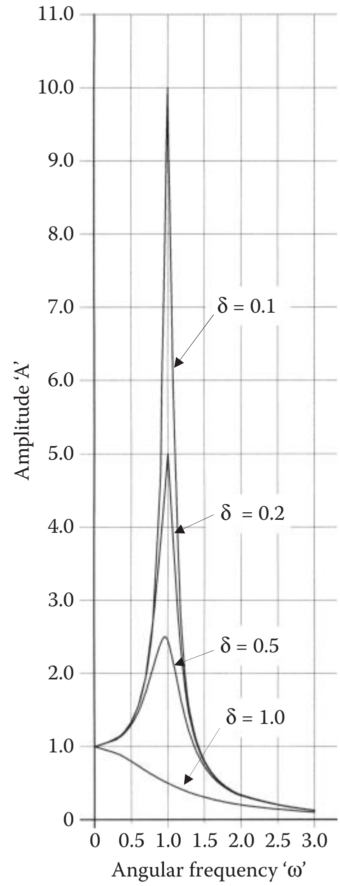


FIGURE 6.41 Amplitude against angular frequency.

EXAMPLE 6.11

A mass–spring–damper system is subjected to a harmonic disturbing force given by the equation F = 400 sin (30 t) N as shown in Figure 6.42. Calculate the amplitude of the mass and the phase angle.

Solution

Given that k = 10,000 N/m, M = 5 kg and c = 150 N · s/m.

$$\omega_n = \sqrt{\frac{k}{M}}$$

$$\omega_n = \sqrt{\frac{10{,}000\ \text{N/m}}{5\ \text{kg}}}$$

$$\omega_n = 44.271\ \text{Hz.}$$

$$c_c = \sqrt{4Mk}$$

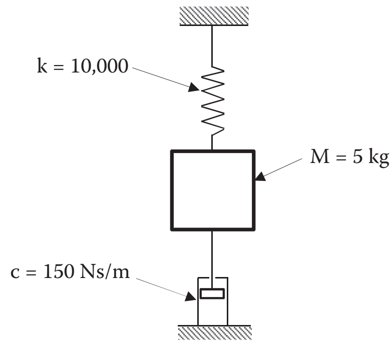


FIGURE 6.42 Example 6.11.

$$c_c = \sqrt{4 \times 5 \text{ kg} \times 10{,}000 \text{ N/m}}$$

$$c_c = 447.214 \text{ kg/s}$$

$$\delta = \frac{c}{c_c}$$

$$\delta = \frac{150 \text{ kg/s}}{447.214 \text{ kg/s}}$$

$$\delta = 0.335$$

From the above equation:

$$(F = 400 \sin(30t) \text{ N})$$

$$F_o = 400 \text{ N}$$

$$\omega = 30 \text{ rad/s}$$

$$A^2 = \left(\frac{F_o}{M}\right)^2 \left\{\frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega \cdot \omega_n)^2}\right\}$$

$$A^2 = \left(\frac{400 \text{ N}}{5 \text{ kg}}\right)^2 \left\{\frac{1}{((44.72 \text{ rad/s})^2 - (30 \text{ rad/s})^2)^2 + (2 \times 0.335 \times 30 \text{ rad/s} \times 44.71 \text{ rad/s})^2}\right\}$$

$$A^2 = 0.003168 \text{ m}^2$$

$$A = 0.056 \text{ m } (56.0 \text{ mm})$$

$$\tan\theta = \frac{2\delta\omega \cdot \omega_n}{\omega_n^2 - \omega^2}$$

$$\tan\theta = \frac{2 \times 0.335 \times 30 \text{ rad/s} \times 44.721 \text{ rad/s}}{44.721 \text{ rad/s}^2 - 30 \text{ rad/s}^2}$$

$$\tan\theta = 0.818$$

$$\theta = 39.283°$$

6.6.4 Harmonic Movement of the Support

Figure 6.39b shows a mass–spring–damper system subject to a base excitation. The mass is restrained to move vertically only and the foundation is subject to a motion described by the equation $y = a \sin(\omega t)$.

It is assumed that the mass will move harmoniously having an amplitude 'A', but it cannot be assumed that the motion of the mass will be in phase with that of the support. Hence, the equation of motion will be amended to $x = A \sin(\omega t + \theta)$ where θ is the phase angle.

During the cycle, the spring will be stretched or shortened by an amount $(x - y)$ at any time. The spring force will be

$$F = k(x - y) \tag{6.104}$$

The three forces acting on the mass are

$$\text{Spring force} = k(x - y) \tag{6.105}$$

$$\text{Damping force} = c\, dx/dt \tag{6.106}$$

$$\text{Inertial force} = Md^2y/dt^2 \tag{6.107}$$

In this instance, there is no external force being directly applied; so balancing the forces gives

$$\theta_t = \theta - \tan^{-1}\left(\frac{F_d}{F_s}\right) \tag{6.108}$$

$$0 = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + k(x - y) \tag{6.109}$$

$$0 = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx - ky \tag{6.110}$$

$$ky = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx \tag{6.111}$$

$$k(a \sin\omega t) = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx \tag{6.112}$$

Comparing this result with Equation 6.99

$$F_o \sin(\omega t) = M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx \tag{6.113}$$

It will be seen that the equations are similar except that the term 'ka' replaces the term F_o. It follows that solutions will be the same with this substitution.

$$A^2 = \left(\frac{ka}{M}\right)^2 \left\{\frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\delta\omega\omega_n\right)^2}\right\} \quad (6.114)$$

$$\tan\theta = \frac{2\delta\omega\omega_n}{\omega_n^2 - \omega^2} \quad (6.115)$$

6.6.5 Magnification Factor

The ratio A/a is also known as the magnification ratio and applies when the support is excited. Equation 6.101 can be rearranged into the following form.

$$\text{M.F.} = \frac{A}{a} = \left(\frac{k}{M}\right)\sqrt{\left\{\frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\delta\omega\omega_n\right)^2}\right\}} \quad (6.116)$$

As $k/M = \omega_n^2$,

$$\text{M.F.} = \sqrt{\left\{\frac{(w_n^2)^2}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\delta\omega\omega_n\right)^2}\right\}} \quad (6.117)$$

$$\text{M.F.} = \sqrt{\left\{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\delta\frac{\omega}{\omega_n}\right)^2}\right\}} \quad (6.118)$$

Equation 6.118 also applies to the case when a harmonic disturbing force is applied as $k_a = F_o$; therefore, it follows that the M.F. will equal the maximum force in the spring F_o.

Figure 6.43 shows the response graph and will be similar to both cases. At low values of 'ω', the support and the mass will move together in synchrony. As the speed is increased and 'ω' approaches 'ω_n', the amplitude 'A/a' increases and the phase angle approaches 90°. As the speed passes through resonance and there is any further increase in speed, the amplitude 'A/a' will reduce and eventually become almost static. The phase angle will tend to 180° at higher speeds. The magnification ratio will be largest at the resonance frequency. As stated before, do not confuse the resonance frequency with the natural frequency.

The maximum magnification factor occurs when

$$\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\delta\frac{\omega}{\omega_n}\right)^2 \quad (6.119)$$

is at a maximum. Using the max and min theory, this expression can be simplified to

$$(1 - r^2)^2 + (2\delta r)^2 \quad (6.120)$$

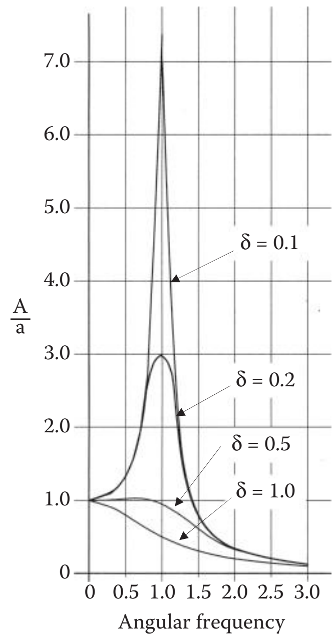


FIGURE 6.43 Response graph.

Differentiating Equation 6.120 w.r.t. r

$$\frac{d\left\{(1-r^2)^2 + (2\delta r)^2\right\}}{dr} = 2(1-r^2)(-2r) + 8\,\delta^2 r \tag{6.121}$$

Equating to zero

$$2(1-r^2)(-2r) + 8\,\delta^2\,r = 0$$

$$r = \sqrt{(1-2\delta^2)} \tag{6.122}$$

Hence, mean peak will occur when

$$\omega = \omega_n \sqrt{(1-2\delta^2)} \tag{6.123}$$

EXAMPLE 6.12

Figure 6.44 shows a mass–spring–damper system where the support is subject to a motion of y = 6 sin(40 t) mm.

Determine the maximum amplitude of the mass and corresponding phase angle.

Solution

Given that k = 10,000 N/m, M = 5 kg and c = 150 N · s/m.

$$\omega_n = \sqrt{\frac{k}{M}}$$

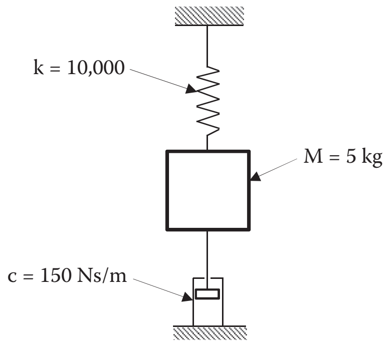


FIGURE 6.44 Example 6.12, mass–spring–damper system.

$$\omega_n = \sqrt{\frac{10{,}000\ \text{N/m}}{5\ \text{kg}}}$$

$$\omega_n = 44.721\ \text{Hz}$$

$$c_c = \sqrt{4Mk}$$

$$c_c = \sqrt{4 \times 5\ \text{kg} \times 10{,}000\ \text{N/ma}}$$

$$c_c = 447.214\ \text{kg/s}$$

$$\delta = \frac{c}{c_c}$$

$$\delta = \frac{150\ \text{kg/s}}{447.214\ \text{kg/s}}$$

$$\delta = 0.335$$

From the equation of motion where a = 6 mm and 40 rad/s.

$$\frac{A}{a} = \left(\frac{k}{M}\right)\sqrt{\frac{1}{(\omega_n^2 - \omega^2)^2 + (2\delta\omega \cdot \omega_n)^2}}$$

$$\frac{A}{a} = \left(\frac{10{,}000}{5}\right)\sqrt{\frac{1}{(44.721^2 - 40^2)^2 + (2 \times 335 \times 40 \times 44.721\ )^2}}$$

$$\frac{A}{a} = 1.581$$

$$A = 1.581 \times a$$

Therefore,

$$A = 9.487\ \text{mm}$$

The phase angle

$$\tan\theta = \frac{2 \times \delta \times \omega \times \omega_n}{\omega_n^2 - \omega^2}$$

$$\tan\theta = \frac{2 \times 0.335 \times 40 \times 44.721}{44.721^2 - 40^2}$$

$$\tan\theta = 2.997$$

$$\theta = 71.545°$$

6.6.6 Transmissibility

When a mass is vibrating on an elastic support, a force is transmitted through the spring and damper to the supporting frame or ground. This force will be the sum of the spring and damping force. Figure 6.45 depicts the corresponding vector diagram.

From the vector diagram, it is deduced that the transmitted force is

$$F_t = \sqrt{(F_s{}^2 + F_d{}^2)} \tag{6.124}$$

Now

$$F_s = k \cdot A$$

$$F_d = c \cdot A \cdot \omega$$

Hence,

$$F_T = \sqrt{[(k \cdot A)^2 + (c \cdot A \cdot \omega)]} \tag{6.125}$$

The transmissibility ratio is defined by F_T/F_o.

The phase angle between the transmitted force and the applied force i

$$\theta_T = \theta - a\tan\left(\frac{F_d}{F_s}\right) \tag{6.126}$$

If the substitution $F_o = ka$ is made to the above work, then it will apply to both harmonic disturbing forces and harmonic motion of the support.

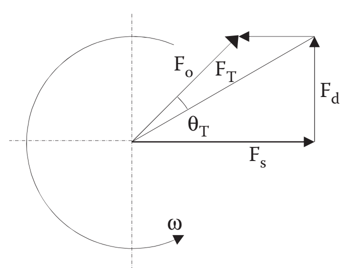


FIGURE 6.45 Vector diagram.

EXAMPLE 6.13

Calculate the transmitted force and the phase angle for Example 6.12.

Solution

Given that $k = 10{,}000$ N/m, $M = 5$ kg, $c = 150$ N · s/m, $\omega = 30$ rad/s, $\omega_n = 44.721$ rad/s and $A = 9.487$ mm, from the equations of motion:

$F_s = k \cdot A$
$F_s = 10{,}000 \text{ N} \times 9.487 \text{ mm}$
$F_s = 94.87 \text{ N}$

$F_d = c \times A \times \omega$
$F_d = 150 \text{ N} \cdot \text{s/m} \times 9.487 \text{ mm} \times 30 \text{ rad/s}$
$F_d = 42.692 \text{ N}$

$F_T = \sqrt{}(F_s^2 + F_d^2)$
$F_T = \sqrt{}[(94.87 \text{ N})^2 + (42.692 \text{ N})^2]$
$F_T = 104.033$ N.

$$\theta_T = \theta - \text{a}\tan\left(\frac{F_d}{F_s}\right)$$

$$\theta_T = 39.283^o - \text{a}\tan\left(\frac{42.692 \text{ N}}{94.87 \text{ N}}\right)$$

$$\theta_T = 15.055^o$$

6.6.7 Using Forced Vibration Response to Measure the Properties of a Structure

The natural frequency and damping coefficient of a structure or component can be measured a number of ways. The simplest one is to attach an accelerometer to a small structure or component and 'striking' it with a 'calibrated' hammer. The hammer is a piece of laboratory equipment and is fitted with an accelerometer in its head. It is used with a 'modal analysis testing equipment'. The accelerometer on the structure is used to measure its response. Another method is to attach the component or small structure it with a vibrating table which inputs a forced vibration into the item. Again, an accelerometer attached to the part measures via the 'modal analyser' and the part's response is measured as the frequency and amplitude of the vibration are increased.

A third method, which is used for very large structures such as bridges, uses an 'exciter' which is essentially an electric motor fitted with an out-of-balance mass. By adjusting the position of the mass on a crank attached to the motor, the vibration amplitude is adjustable and the speed of the motor will vary the frequency.

In the case of measuring the frequency response of the bridge, the exciter is placed at a point mid-span. A number of accelerometers are then placed at various points along the structure at measured distances from the position of the exciter with one accelerometer positioned close to the exciter. Measurements from the accelerometers are recorded for various frequencies and amplitudes.

A graph similar to the one shown in Figure 6.46 is plotted from the results. The maximum response 'X_{max}' is measured and a line is drawn at the amplitude $X_{max}/\sqrt{2}$. The frequencies ω_1 and ω_2 together with ω_{max} are then measured as shown in the figure.

The bandwidth of the response ($\Delta\omega$) is defined as

$$\Delta\omega = \omega_2 - \omega_1 \tag{6.125}$$

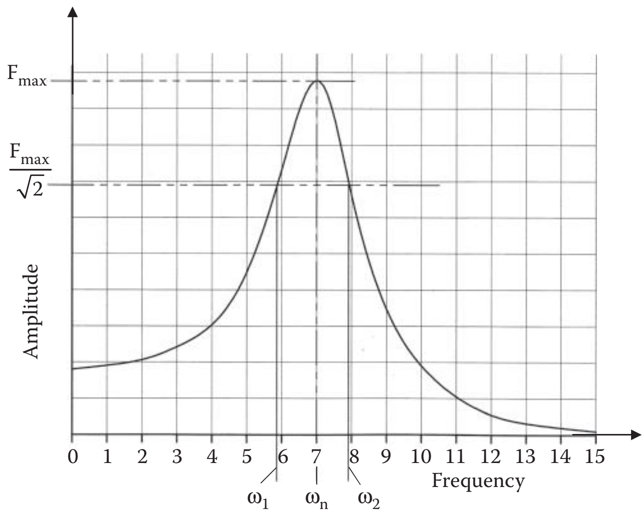


FIGURE 6.46 Bandwidth.

$$A_o = \frac{1 \times 10^{-4} \times 500}{\left[\left(1 - \frac{[\pi^2]}{2.236^2}\right)^2 + \left(2 \times 0.224 \times \frac{\pi}{2.236}\right)^2\right]^{0.5}}$$

Like the logarithmic decrement, the bandwidth is a measure of the damping that exists in the system.

The natural frequency and damping coefficient can be estimated using the following formula

$$\delta \approx \frac{\Delta\omega}{2\omega_{max}} \tag{6.126}$$

This formula is accurate for small δ – say δ > 0.2.

From the study of steady state response, it can be shown that

$$\omega_{max} = \omega_n \sqrt{1 - 2\delta^2} \tag{6.127}$$

and for small δ

$$\omega_{max} \approx \omega_n \tag{6.128}$$

The next stage is to develop an expression relating bandwidth Δω to δ. The frequencies ω_1 and ω_2 are calculated next. The maximum amplitude of vibration is calculated by setting $\omega = \omega_n$, which gives

$$A_{max} = \frac{K \cdot F_o}{2\delta\sqrt{1 - \delta^2}} \tag{6.129}$$

where K = 1/k.

At the two frequencies of interest, it is known that $A_o = A_{max}/\sqrt{2}$; hence, ω_1 and ω_2 have to be the solutions of the equation:

$$\frac{KF_o}{2\delta\sqrt{1-\delta^2}} \cdot \frac{1}{\sqrt{2}} - \frac{KF_o}{\sqrt{\left(1-\frac{\omega^2}{\omega_n{}^2}\right)^2 - (2\delta\omega\omega_n)^2}} \tag{6.130}$$

Rearranging:

$$\omega^4 \frac{1}{2} 2\omega^2\omega_n{}^2(1-2\delta^2) - \omega_n{}^4 - 8\delta^2\omega_n{}^4(1-\delta^2) = 0 \tag{6.131}$$

This produces a quadratic equation for ω^2 and has the solutions

$$\omega_1 = \left\{\omega_n{}^2(1-2\delta^2) - 2\omega_n 2\delta\sqrt{1-\delta^2}\right\}^{1/2} \tag{6.132}$$

$$\omega_2 = \left\{\omega_n{}^2(1-2\delta^2) + 2\omega_n 2\delta\sqrt{1-\delta^2}\right\}^{1/2} \tag{6.133}$$

Expanding both expressions in a Taylor series

$$\omega_1 \approx \omega_n(1-\delta) \tag{6.134}$$

$$\omega_2 \approx \omega_n(1+\delta) \tag{6.135}$$

Finally,

$$\Delta\omega = \omega_2 - \omega_1 \tag{6.136}$$

$$\Delta\omega = 2\delta\omega_n \tag{6.137}$$

EXAMPLE 6.14

Consider an idealised spring–mass–damper structure having a stiffness of 10 kN/m, a mass of 2000 kg and a damping coefficient of 2 kN · s. The structure is subject to a harmonic force of 500 N at a frequency of 0.5 Hz.

Calculate the steady state amplitude of vibration.

Solution

Given that $k = 10$ kN/m, $M = 2000$ kg, $c = 2$ kN · s/m, $F = 500$ N, frequency = 0.5 Hz, $A = 500$ N, $\omega = 0.5 \times 2\pi$, $\omega = \pi$ rad/s:

$$\omega_n = \sqrt{\frac{k}{M}}$$

$$\omega_n = \sqrt{\frac{10\ \text{kN/m}}{2000\ \text{kg}}}$$

$$\omega_n = 2.236\ \text{rad/s}$$

$$\delta = \frac{c}{2 \cdot \sqrt{k \cdot M}}$$

$$\delta = \frac{24\ \text{kN/m}}{2 \cdot \sqrt{10\ \text{kN/m} \times 2000\ \text{kg}}}$$

$$\delta = 0.224$$

$$K = \frac{1}{k}$$

Steady state amplitude:

$$A_o = \frac{K \times F}{\left[(1 - (\omega^2/\omega_n{}^2))^2 + (2 \cdot \delta \cdot (\omega/\omega_n))^2\right]^{0.5}}$$

$$A_o = \frac{1 \times 10^{-4} \times 500}{\left[\left(1 - \frac{[\pi^2]}{2.236^2}\right)^2 + \left(2 \times 0.224 \times \frac{\pi}{2.236}\right)^2\right]^{0.5}}$$

$$A_o = 43.14\ \text{mm}$$

7 Introduction to Control Systems Modelling

7.1 INTRODUCTION

Automatic control of machines and processes is a fundamental to the successful performance of modern industry.

Although an automatic control system was first developed by the Egyptians in the third century B.C. with the Ktesibios water clock in Alexandria using a form of a feedback control device, this field was largely left alone until 1620 using a closed-loop feedback control for a furnace which is attributed to Drebbel and 1788 with the design of a centrifugal flyball governor developed by James Watt to regulate the speed of steam engines being built by Boulton and Watt (Figure 7.1).

In 1868, J.C. Maxwell in his paper 'On Governors' studied the instability of the flyball governor using differential equations. This was one of the first papers to use mathematics to describe a control system and demonstrate the importance of mathematical models to explain the complex phenomena. This signalled the beginning of mathematical control and systems theory. Parts of the control theory had appeared earlier but Maxwell's paper was the first to bring it all together.

Significant developments were made in control theory in the following 100 years in developing new mathematical techniques which made it then possible to control more complex dynamic processes. These developments included optimal control methods in the 1970s and 1980s which have led to safer and more efficient aircraft travel and automotive engines together with more efficient (and safer) chemical processes, and the list is expanding.

Modern-day control engineering (also known as control systems engineering) is a relatively new field of study that has gained significant attention in other areas than mechanical and electrical engineering such as financial and biological processes being amenable to control techniques.

The function of any control system is to automatically regulate the output of a system and maintain it at a desired value. The desired value is the input to the system. If the input is changed, the output must respond to the new set value.

There are some basic properties and terminology that are used to model a control system and these will be discussed more fully later in this chapter.

7.1.1 Basics of Control Theory

In this section, the reader will be introduced to the basic underlying theory of control systems and the terminology used.

To start, there are two general classifications of control systems:

1. Open-loop systems where the control action is independent of the output from the system.
2. Closed-loop systems in which the control action is dependent on the output.

7.1.2 Open-Loop Control System

An open-loop system requires an independent external action to ensure the required output. As an example, human intervention is needed when a saucepan is boiling on a heating ring to switch it off to prevent it boiling dry.

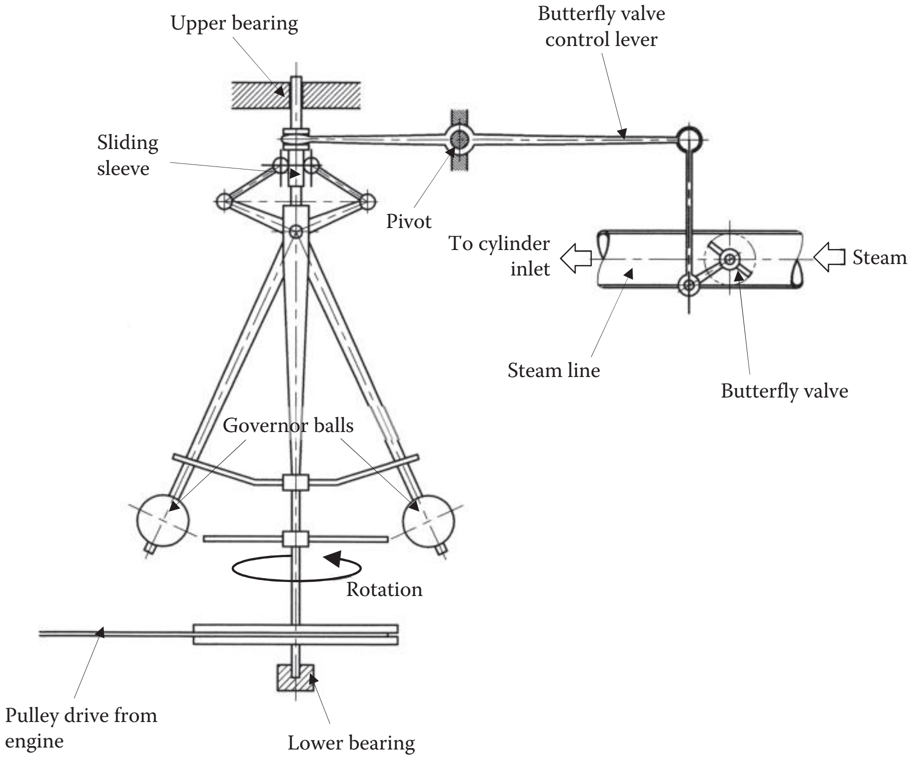


FIGURE 7.1 Centrifugal flyball governor.

Open-loop systems have two important features:

- They depend on their calibration for accurate operation.
- Open-loop systems are not normally affected by stability problems; there is little risk that the input will result in an unexpected output.

Figure 7.2 gives an example of an open-loop system.

7.1.3 Closed-Loop Control System

Closed-loop systems are also referred to as 'feedback' systems where the output is compared to the input into the system. These types of systems are complicated, requiring the use of differential equations for their solution.

Figure 7.3 illustrates a typical closed-loop control system using a feedback loop.

7.1.4 Control System Definitions

The following are some basic control system definitions.

7.1.4.1 System

A system is a collection of entities that form and act as a single unit.

7.1.4.2 Input

A signal supplied from an external source to produce a specified response from the control system.

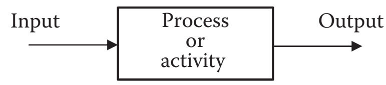


FIGURE 7.2 Example of an open-loop system.

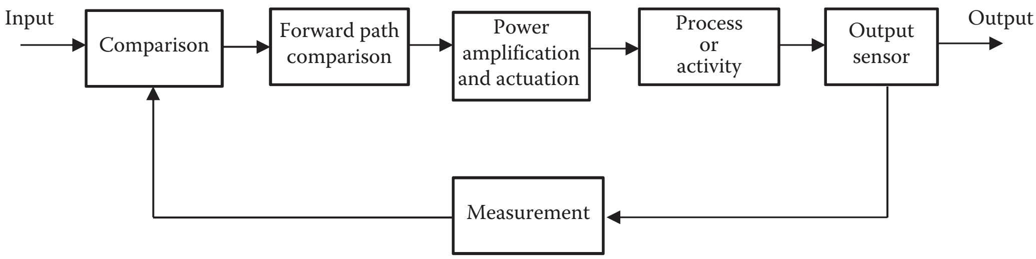


FIGURE 7.3 Example of a closed-loop system.

7.1.4.3 Output

The output is the actual response from the control system. It may not be equal to the response implied by the input signal.

7.1.4.4 Open Loop

An open-loop control system is one where the control action is independent of the output.

7.1.4.5 Closed Loop

A closed-loop control system is one where the control action is related to the input.

7.1.4.6 Feedback

Feedback is the property of a closed-loop control system where the output signal is compared with the input signal to enable an appropriate control action that may be completed in accordance with the requirements of the control system.

7.1.4.7 Servomechanism

This is a power-amplifying feedback control system in which the controlled variable is either a mechanical position or a time derivative of position such as velocity or acceleration.

7.1.4.8 Regulator

A feedback control system where the reference input is fixed over the operating time period. The primary function of the regulator is to maintain a constant output signal.

7.1.5 Feedback Characteristics

A feedback in a control system results in the following advantages:

1. Increased accuracy: the output can be made to reproduce the input.
2. Reduced sensitivity to system characteristics.
3. Reduction in the effects of non-linearities.
4. Increased bandwidth: the system can be made to respond to a wider range of frequencies.
5. The major disadvantage of using feedback is the increased risk of instability and the cost of implementation.

7.1.6 Control Models

The study of control systems requires a good working knowledge of

1. Differential equations and other mathematical techniques
2. Block diagrams and transfer functions
3. Signal flow diagrams

Block diagrams and signal flow paths are shorthand representations used in the construction of schematic diagrams representing the physical system, or the set of mathematical equations that characterise the individual component parts of the control system.

7.1.7 Block Diagrams and Transfer Functions

Any item in any system may be represented by a simple block with arrows representing the direction flow of the information or signal. A block with an input and associated output is shown in Figure 7.4. The block usually contains a description or name of the element or the mathematical operation to be accomplished. In this example, the block represents the function

$$G(s) = \frac{x(s)}{F(s)} = \frac{1}{k} = C \tag{7.1}$$

In general terms, the input is designated as 'θ_i' and the output as 'θ_o'. It may also be seen as 'θ_1' and 'θ_2', respectively. The ratio of the output to the input is mostly shown as $G = \theta_o/\theta_i$. When the model is a differential equation, the Laplace transform is used which introduces the complex operator 's'. In this case, the 'G' is called the 'transform function' and is written as

$$G(s) = \frac{\theta_o(s)}{\theta_i(s)} \tag{7.2}$$

A generalised feedback control system is shown in Figure 7.5.

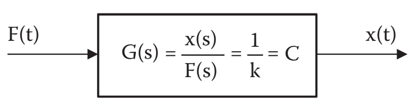


FIGURE 7.4 Block diagram.

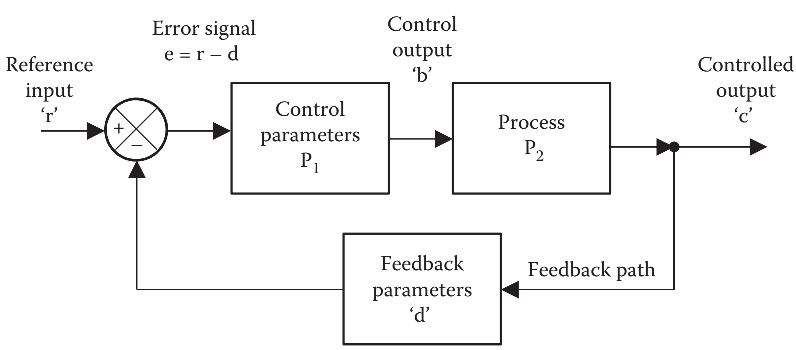


FIGURE 7.5 A generalised feedback control system.

7.2 ENGINEERING SYSTEM MODELS

When studying control systems, it will be surprising to see the similarity there is between the various branches of engineering. The reader will need to have a broad base of knowledge of engineering science to understand the various elements and see how many of them are mathematically similar to each other.

In this section, different kinds of systems will be reviewed to see how they conform to similar laws with clear analogies between them including

- Mechanical
- Electrical
- Thermal
- Fluids

The fundamental laws which are used mostly include

- Resistance (R)
- Capacitance (C)
- Inductance (L)
- Conservation laws

Table 7.1 shows the equivalent components between the various systems. It will be useful to note that capacitance is a zero-order differential equation, and resistance will be a first-order differential equation, where as inertia and inductance are second-order differential equations.

7.2.1 Similarities of Elements between Systems

7.2.1.1 Capacitance

The symbol 'C' is used in electrical, thermal and fluid capacitance. Capacitance in mechanical systems is equivalent to 1/k, where k is the spring stiffness.

TABLE 7.1
Comparison of Various Quantities between Systems

Mechanical	Fluid	Thermal	Electrical
Spring: $x = C \cdot F = (1/k) \cdot F$	Fluid capacitor: $M = C \cdot \Delta p$	Thermal capacitor: $Q = C\Delta T$	Electrical capacitor: $Q = C \cdot V$
Damper: Force = kd × velocity $F = kd\ dx/dy$ Torque = kd × angular velocity	Fluid friction laws do not conform	Heat transfer laws: $\Delta T = R\Phi$ $\Delta T = R\ dQ/dt$	Ohm's law $V = I \cdot R$ $V = R\ dQ/dt$
Newtons second law of motion: Force = mass × acceleration $F = M\ d^2x/dt^2$	Fluid interance: $\Delta p = L\ d^2v/dt^2$	No equivalence	Law of inductors: $V = L\ d^2q/dt^2$
D'Alembert's principles: ΣForce = 0 ΣMoment = 0	Laws of conservation of mass: ΣMass = constant	Laws of conservation of energy: ΣEnergy = constant	Kirchoff's law: ΣCurrent = 0

7.2.1.2 Resistance

In electrical and thermal systems, the symbol 'R' is used for resistance.

7.2.1.3 Inductance, Inertia and Inertance

'L' is the symbol used for electrical inductance and fluid intertance. In mechanical systems, mass is an equivalent property used in linear motion and moment of inertia is used for angular motion.

7.2.1.4 Other Symbols Used

Electrical charge and the quantity of heat are symbolised by 'Q' and this is also equivalent to displacement in mechanical systems such as distance (usually 'x') and angle (usually 'θ').

'V' is recognised as the symbol for electrical voltage (potential difference or e.m.f.) and has its equivalent to temperature in thermal systems. The force 'F' in mechanical systems and 'p' for pressure in fluid systems are used.

In electrical systems, 'I' or 'i' is the symbol used in electrical systems for electrical current, where the symbol 'Φ' is used to symbolise heat flow rate and 'v' or 'u' is the symbol used for velocity in mechanical systems.

7.2.2 Laplace Transforms

Laplace transforms have been covered in some detail in Chapter 1, Mathematics and will be touched upon in later sections in this chapter.

To recap: the use of Laplace transforms is to allow differential equations to be converted into a normal algebraic equation where the quantity 's' is a normal algebraic quantity. It is considered a shorthand method for writing differential coefficients, that is,

$$\frac{d\theta}{dt} \text{ can be written as } s\theta$$

$$\frac{d^2\theta}{dt^2} \text{ becomes } s^2\theta$$

and

$$\frac{d^n\theta}{dt^n} \text{ becomes } s^n\theta$$

7.2.3 Transfer Functions

Models of systems can often be written in the form of the ratio of output/input and if the model is turned into a function of 's', it is called a transfer function and will usually be represented as G(s).

That is,

$$G(s) = \frac{x}{F}s = \frac{1/k}{s^2(M/k) + s(k_d/k) + 1}$$

In the next section, mathematical models of some basic mechanical systems will be considered.

7.2.4 Linear Mechanical Systems

7.2.4.1 Spring

The basic law of a mechanical spring (either helical or leaf) is force ∞ change in length. Figure 7.6 shows the model with the mechanical symbols and with it is depicted as a block diagram in Figure 7.7.

This relationship has no derivatives in that it may be written as a function of either 't' or 's' with no transform involved.

- As a function of time, it can be written as F(s) = kx(s).
- This equation can be rearranged as a transfer function such that (x/F)(s) = 1/k = C.

Where C is the reciprocal of stiffness and is referred to as the mechanical capacitance. The use of 'k' is the preferred symbol in mechanics but 'C' is mostly used as it is directly analogous to electrical capacitance.

7.2.4.2 Damper or Dashpot

A damper may be characterised as a piston within a cylinder and moves in a viscous fluid; the force is directly proportional to the velocity of the piston.

$$F \infty v \quad \text{where v is the first derivative of distance}$$

This equation can be written as

$$F \propto \frac{dx}{dt}$$

The basic law of a damper is

$$F_t = k_d \frac{dx}{dt} \quad \text{where } k_d \text{ is the damping coefficient}$$

When changed into Laplace form

$$F = k_d\, s\, x$$

Rearranged into a transfer function

$$\frac{x}{F}(s) = \frac{1}{k_d s}$$

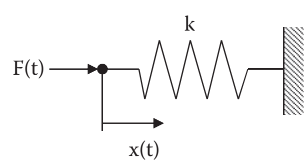


FIGURE 7.6 Model of a spring.

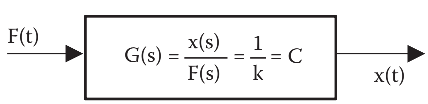


FIGURE 7.7 Block diagram representation for a spring.

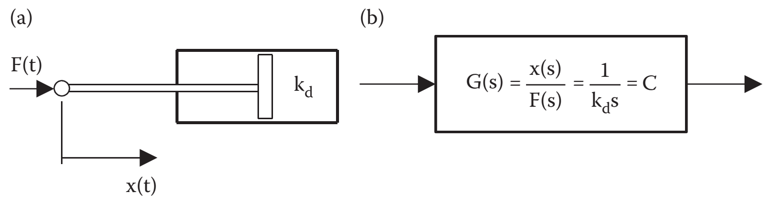


FIGURE 7.8 Damper symbol. (a) Schematic for a damper system and (b) the block model for a damper.

k_d is the damping coefficient which has units of force/velocity or N s/m. Figure 7.8 shows the model as a mechanical symbol and its equivalent control block.

7.2.4.3 Mass

Newton's second law of motion states that a force is required to accelerate a mass and is written as force = mass × acceleration.

Acceleration is the second derivative of 'x' with time.

The basic law is

$$F(t) = M\frac{d^2x}{dt^2}$$

and when changed into Laplace form

$$F = Ms^2x$$

and rearranged into a transfer function

$$\frac{x}{F}(s) = \frac{1}{Ms^2}$$

Figure 7.9 depicts the mass under the influence of a force and its equivalent control block.

7.2.4.4 Mass–Spring System

In Chapter 6 on vibrations, the spring–mass system was introduced in which motion only occurs in one direction; hence, the system has only one degree of freedom. It is normal for the direction of motion to be expressed as the 'x' direction, regardless of the actual direction.

Figure 7.10a depicts the spring–mass system and Figure 7.10b the free body diagram. The input is a disturbing force 'F' which will be a function of time F(t). This disturbing force could be a sinusoidal force. The output is a motion 'x' which will be a function of time x(t).

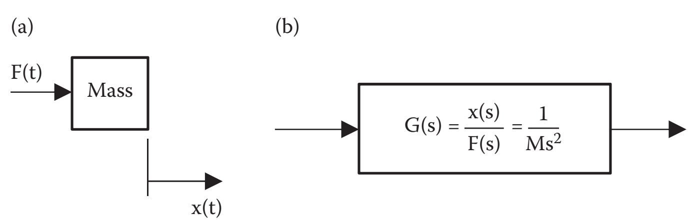


FIGURE 7.9 Block diagram of a mass subject to acceleration. (a) The free body diagram for a mass subject to acceleration and (b) the equivalent block diagram.

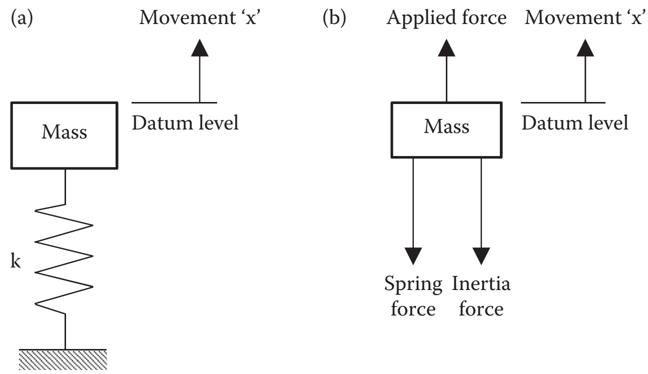


FIGURE 7.10 (a) Physical system and (b) free body diagram for a mass–spring system.

Now, let 'x' be a positive direction vertically; the input force is opposed by the spring force and inertia force which will always oppose any changes to motion as stated in Newton's third law of motion.

Hence,

$$\text{Spring force} = kx$$

$$\text{Inertia force} = M\frac{d^2x}{dt^2}$$

D'Alambert's principle states that all forces and moments on a body should equate to zero, which in this case means

$$F(t) - kx(t) - M\frac{d^2x}{dt^2(t)} = 0$$

or

$$F(t) = M\frac{d^2x}{dt^2(t)} + kx(t)$$

Changing to a function of 's'

$$F(t) = Ms^2\,x + kx$$

$$= x[Ms^2 + k]$$

$$x(s) = \frac{F}{Ms^2 + k}$$

$$= \frac{F(1/M)}{s^2 + k/M}$$

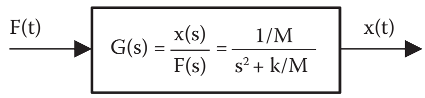


FIGURE 7.11 Block diagram for a mass–spring system.

This may be shown as a transfer function

$$G(s) = \frac{x(s)}{F(s)}$$

$$= \frac{1/M}{s^2 + k/M}$$

Figure 7.11 shows the block diagram.

7.2.4.5 Spring–Damper System

Consider Figure 7.12a.

Force balance as a function of time

$$F(t) = kx + kd\frac{dx}{dt}$$

Force balance as a function of 's'

$$F(s) = kx + k_d \cdot sx$$

Rearranging as a transfer function

$$\frac{x}{F}(s) = \frac{1/k}{(k_d/k)s + 1}$$

The units of k_d/k are seconds and this is the time constant for the damped system

$$T = \frac{k_d}{k}$$

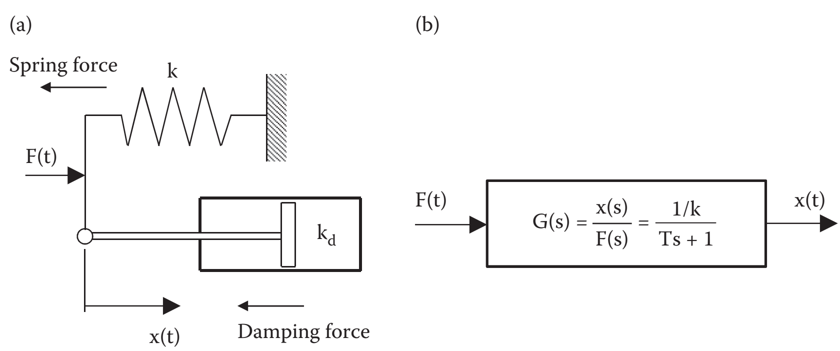


FIGURE 7.12 System (a) and block model (b) for a spring–damper.

The standard first-order equation

$$\frac{x}{F}(s) = \frac{1/k}{Ts+1}$$

This result will be studied a number of times in the following sections. The block diagram is shown in Figure 7.12b.

7.2.4.6 Mass–Spring–Damper System

Figure 7.13a depicts a mass–spring–damper system in which a force ‘F’ is applied to the system resulting in a displacement ‘x’, both being a function of time ‘t’.

Hence,

$$\text{Spring force} \quad F_s = kx$$

$$\text{Damping force} \quad F_d = k_d \frac{dx}{dt}$$

$$\text{Inertia force} \quad F_i = M\frac{d^2x}{dt^2}$$

These three forces oppose the motion of the system; so if the total force acting on the system is zero, then

$$F = F_i + F_d + F_s$$

$$F(t) = M\frac{d^2x}{dt^2} + k_d\frac{dx}{dt} kx$$

$$F(s) = Ms^2 x + k_d sx + kx$$

$$G(s) = \frac{x}{F}s = \frac{1/k}{s^2(M/k) + s(k_d/k) + 1}$$

An examination of the units of $(M/k)^{0.5}$ is in seconds and this is the second-order time constant with the symbol ‘T’. Therefore, the transfer function can be written as

$$G(s) = \frac{x}{F}(s) = \frac{1/k}{T^2s^2 + s\delta Ts + 1}$$

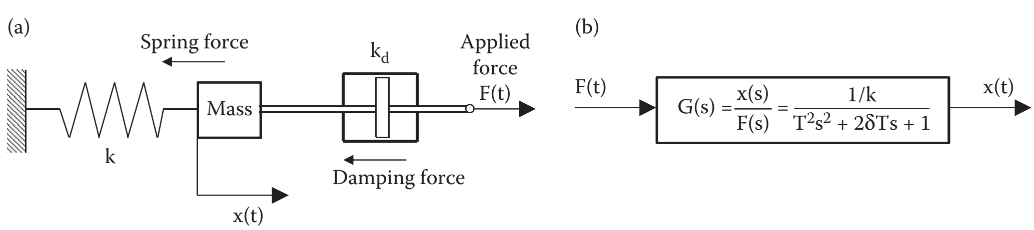


FIGURE 7.13 (a) Schematic for a spring–mass–damper. (b) Block diagram for a spring–mass–damper.

where δ is the damping ratio which is defined as

$$\delta = \frac{k_d}{C_c} T$$

where C_c is the critical damping ratio defined as $(4Mk)^{0.5}$.

The term 2δT is

$$2\frac{k_d}{C_c}T = \frac{2kd\sqrt{M/k}}{\sqrt{4Mk}}$$

$$= \frac{2kd\sqrt{M}}{2\sqrt{M\sqrt{k}\sqrt{k}}}$$

$$= \frac{k_d}{k}$$

Therefore, the foregoing is correct.

Figure 7.13b depicts the block diagram for the transfer function for this type of system. It is a second-order transfer function and an analysis of it will be dealt with in more detail later.

EXAMPLE 7.1

A spring–mass–damper system has the following values:

Stiffness (k) = 1000.0 N, mass (M) = 5.0 kg, damping coefficient (k_d) = 30.0 Ns/m

1. Calculate:
   a. The time constant
   b. The critical damping coefficient
   c. The damping ratio
2. Derive the equation for the force required when the mass is accelerating.
3. Using the derived equation, evaluate the static deflection when the force 'F' = 15.0 N.
4. Using the equation, evaluate the force required to accelerate the mass at 6.0 m/s^2 when the velocity is 0.70 m/s.

Solution

$$k = 1000.0 \text{ N}$$

$$M = 5.0 \text{ kg}$$

$$k_d = 30.0 \text{ Ns/m}$$

1.

$$T = \sqrt{\frac{M}{k}} = \sqrt{\frac{5.0}{1000.0}} = 0.0707 \text{ s}$$

$$C_c = \sqrt{4Mk} = \sqrt{4.5 \cdot 1000} = 141.421 \text{ Ns/m}$$

$$\delta = \frac{k_d}{C_c} = \frac{30}{141.421} = 0.212$$

2. For a constant acceleration $S^2X = a$ (acceleration) and $SX = V$ (velocity)

$$F = k \cdot X(T^2\ S^2 + 2\delta T \cdot S + 1)$$

$$F = 1000 \times X(0.0707^2\ S^2 + 2.0 \times 212.0 \times 0707\ S + 1)$$

$$F = X(5.1409\ S^2 + 29.9768\ S + 1000)$$

$$F = 5.1409\ S^2X + 29.9768\ SX + 1000X$$

$$F = 5.1409\ a + 29.9768\ V + 1000X$$

3. For a constant force and a static position, there is neither acceleration nor velocity; therefore, the S^2 and S terms are zero.
   Consider the deflection 'X' when the force 'F' is 15.0 N

$$\frac{F}{X} = 1000\,\text{N}, \quad X = \frac{15}{1000} = 0.015\,\text{m}\ (15.0\,\text{mm})$$

4. Force needed to accelerate the mass at 6.0 m/s^2 when the velocity is 0.7 m/s

$$a = 6.0\ \text{m/s}^2; \quad V = 0.70\ \text{m/s}.$$

$$F = (5.1409 \times 6.0) + (29.9768 \times 0.70) + 1000\ X$$

$$F = 30.8454 + 20.9838 + 1000\ X$$

$$F = 51.8292 + 1000\ X$$

The deflection 'X' will need to be evaluated from $x = V^2/2a$, which in this case yields 0.0408 m (4.08 mm)

$$F = 92.663\ \text{N}$$

7.2.5 Rotary Mechanical Systems

The following work is essentially the rotary equivalent of the previous work.

7.2.5.1 Torsion Bar

A metal rod which is restrained at one end and supports a mass at the other is equivalent to the mass and spring (Figure 7.14). If a torque is applied to the rod, a twist will result in the rod. The applied torque is directly proportional to the angle of twist and the ratio T/θ is the torsional stiffness of the rod and is denoted as 'k' as in the linear systems.

The nomenclature used is

T	Torque	Nm
θ	Angle of twist	radians
k	Torsional stiffness	Nm/rad

Equating the torques $T(t) = k\theta\ (t)$
Change to Laplace form $T(s) = k\theta\ (s)$
Write as a transfer function $\theta/T(s) = 1/k$

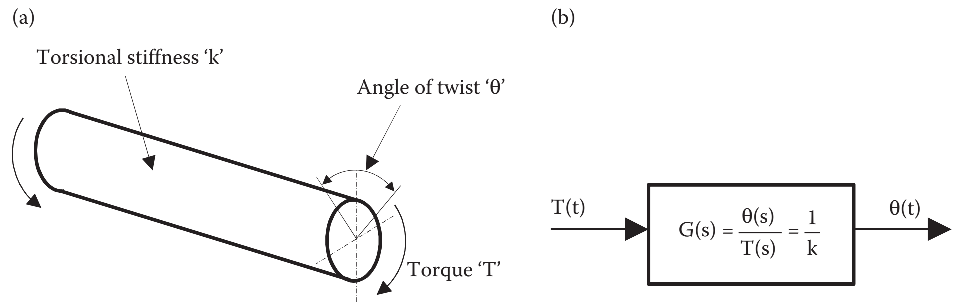


FIGURE 7.14 (a) and (b): Schematic for a torsion bar.

7.2.5.2 Torsion Damper

Rather like the dampers used in the linear systems which characterise a piston operating in an enclosed tube filled with a viscous fluid, the rotary dampers may be idealised using vanes also rotating in a viscous fluid where the torque required to rotate the vanes is directly proportional to the angular velocity. k_d is the torsion damping coefficient with values of Nm s/radian.

Hence,

$$T(t) = k_d \frac{d\theta}{dt}, \quad T(s) = k_d s\theta, \quad G(s) = \frac{\theta}{T}(s) = \frac{1}{k_d s}$$

Figure 7.15a depicts a representation of the torsional damper and Figure 7.15b shows the equivalent block diagram.

7.2.5.3 Moment of Inertia

A rotating mass will oppose the change to rotary motion and Newton's second law for rotating masses may be represented by $T = I\,d^2\theta/dt^2$, where I is the moment of inertia in kg m^2.

$$T = I\frac{d^2\theta}{dt^2} \quad T(s) = Is^2\theta \quad G(s) = \frac{\theta}{T}(s) = \frac{1}{Is^2}$$

Note: The symbol 'J' is also used to represent the moment of inertia.

Figure 7.16a depicts a torsion bar supporting a mass subject to an applied torque and Figure 7.16b shows the equivalent block diagram.

7.2.5.4 Geared Systems

Geared system is a common element that is found in many mechanical systems where an inertial mass is driven through a gear system via a drive motor, whereas the effect of the inertia is significantly altered when the inertia is referred back to the motor.

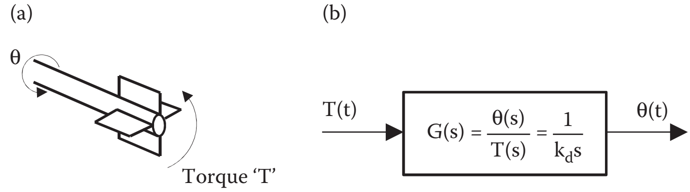


FIGURE 7.15 (a) and (b): Schematic for a torsion damper.

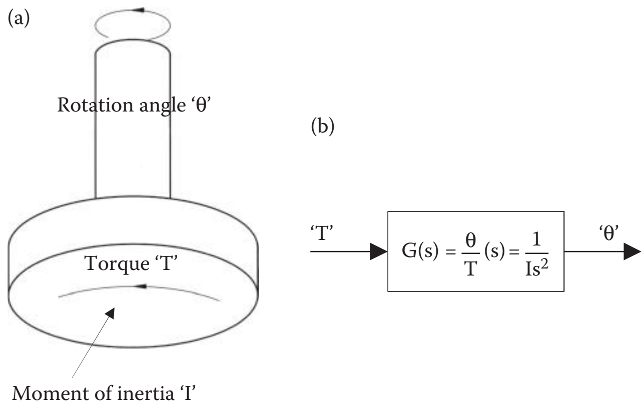


FIGURE 7.16 (a) and (b): Torsion bar supporting a mass subject to an applied torque.

Consider a motor coupled to a load through a set of gears as shown in Figure 7.17. The two bearings provide damping in the system (viscous friction).

θ_i is the input rotation provided in this case by the motor and θ_o is the output rotation.

The gear ratio is $G_r = \theta_o/\theta_i$. As this is a fixed ratio and is not a function of time, the speed and acceleration will also be in the same ratio.

Therefore,

$$\frac{d\theta_i}{dt} = \omega_i \quad \frac{d\theta_o}{dt} = \omega_o \quad G_r = \frac{\omega_o}{\omega_i} \quad \omega \text{ is the angular velocity}$$

$$\frac{d^2\theta_i}{dt^2} = \alpha_i \quad \frac{d^2\theta_o}{dt^2} = \alpha_o \quad G_r = \frac{\alpha_o}{\alpha_i} \quad \alpha \text{ is the angular acceleration}$$

The power that is transmitted by a shaft is given by Power = ωT. Where no power is lost, the output and input powers must be equal; hence, it follows:

$$\omega_i T_i = \omega_o T_o \quad \text{therefore}$$

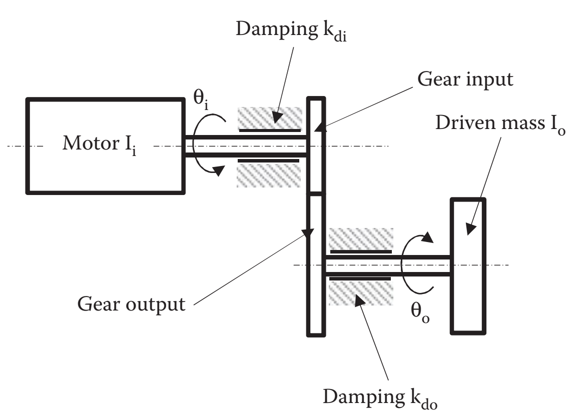


FIGURE 7.17 Geared system.

$$T_i = \frac{\omega_o T_o}{\omega_i}$$

$$= G_r T_o$$

In practice, friction will have a significant effect on torque. The inertia torque due to the inertia on the output shaft I_o

$$T_o = I_o \alpha_o$$
$$= I_o \alpha \cdot G_r$$

Torque on the input shaft

$$T_i = T_o \cdot G_r$$
$$= I_o \alpha_i \cdot G_r^2$$

The damping torque on the output shaft

$$T_o = k_{do} \cdot \omega_o$$

$$= k_{do} \cdot \omega_i \cdot G_r$$

The damping torque on the input shaft

$$T_i = T_o \cdot G_r$$
$$= k_{do} \cdot \omega_i \cdot G_r^2$$

Now considering that there is an inertia and damping torque on the input shaft and on the output shaft, the total torque produced on the input shaft:

$$T_i = I_i \cdot \alpha_i + k_{di} \cdot \omega_i + G_r \cdot T_o$$

$$T_o = I_o \cdot \alpha_o + k_{do} \omega_o$$

$$T_i = I_i \cdot \alpha_i + k_{di} \cdot \omega_i + G_r \{I_o \cdot \alpha_o + k_{do} \cdot \omega_o\}$$

$$T_i = I_i \cdot \alpha_i + k_{di} \cdot \omega_i + G_r^2 \cdot I_o \cdot a_i + G_r^2 k_{do} \cdot \omega_i$$

$$T_i = \alpha_i \cdot (I_i + G_r^2 \cdot I_o) + \omega_i \cdot (k_{di} + G_r^2 \cdot k_{do})$$

Now $(I_i + G_r^2 \cdot I_o)$ is the effective moment of inertia I_e, $(k_{di} + G_r^2 \cdot k_{do})$ is the effective damping coefficient k_{de}.

The equation may be written

$$T_i = \alpha_i (I_e) + \omega_i (k_{de})$$

and in calculus form this can be written as

$$T_i = \frac{d^2\theta}{dt^2}(I_e) + \frac{d\theta}{dt}(k_{de})$$

Changing this equation into a function of 's'

$$T_i(s) = s^2\theta(I_e) + s_\theta(k_{de})$$

$$= s\theta\,(sI_e + k_{de})$$

The output is the angular rotation of the input shaft (motor) and the input is the input torque, whereas the geared system can be represented as a transfer function.

$$\frac{\theta(s)}{T_i(s)} = \frac{1/I_e}{s}\left(s + \frac{k_{de}}{I_e}\right)$$

Figure 7.18 represents the above transfer function as a block diagram.

EXAMPLE 7.2

It is required to rotate a Radar aerial through a geared reduction using a DC servo motor.
The system parameters are

- Gear reduction ratio is 10:1.
- The servo motor has a moment of inertia of 0.5 kg · m^2.
- The driven mass has a moment of inertia of 1.2 kg m^2.
- The damping on the motor is 0.1 Nm · s/rad.
- The damping on the aerial bearings is 0.05 Nm · s/rad.

Determine the transfer function θ/T_m in its simplest form and calculate the motor torque required to

1. Turn the aerial at a constant rate of 0.02 rad/s.
2. Accelerate the aerial at 0.005 rad/s^2 when $\omega = 0$.

Solution

1.

$$I_e = (I_i + G_r^2 \cdot I_o)$$
$$= (0.5 + 10^2 \times 1.2)$$
$$= 120.5\ \text{kg} \cdot \text{m}^2$$

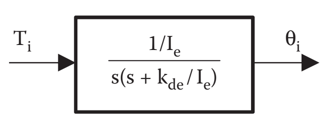


FIGURE 7.18 Block diagram for a geared example.

$$\begin{aligned} k_{de} &= (k_{di} + G_r^2 \cdot k_{do}) \\ &= (0.1 + 10^2 \times 0.05) \\ &= 5.1\,\text{Nm s/rad} \end{aligned}$$

$$\frac{\theta(s)}{T_i(s)} = \frac{(1/I_e)}{s}\left\{s + \frac{k_{de}}{I_e}\right\}$$

$$\frac{\theta}{T_i}(s) = \frac{1/I_e}{s(s + k_{de}/I_e)}$$

$$T_i = I_e \cdot \alpha + k_{de} \cdot \omega$$

$$T_i = 120.5\alpha + 5.1\omega$$

If the aerial is moving at a constant speed, α is zero. Therefore

$$\begin{aligned} T_i &= 5.1\omega \\ &= 5.1 \times 0.02 \\ &= 0.102\,\text{Nm} \end{aligned}$$

2. When the aerial is accelerating at 0.005 rad/s^2, the input (motor) acceleration will be 10 times larger (0.005 rad/s^2 × 10) = 0.05 rad/s^2.

$$Ti = 120.5\alpha + 5.1\omega$$

$$= 60.25\ \text{Nm} \quad \text{when } \omega = 0$$

7.2.6 Thermal Systems

7.2.6.1 Heating and Cooling

If a mass (M) is heated by immersing in a bath of hot fluid (as shown in Figure 7.19), the quantity of heat (Q) absorbed by the mass will be a function of the specific heat capacity (C) multiplied by the change in temperature from ambient (T_1) to the final temperature of the mass (T_2).

This can be expressed in algebraic form

$$Q = Mc(T_2 - T_1)$$

From the laws of heat transference

$$\begin{aligned} dQ &= Mc\ d\theta_1 \\ &= C \cdot d\theta_1 \end{aligned}$$

where C = Mc and is the thermal capacitance expressed in joules/kelvin.

Dividing both sides by dt

$$\frac{dQ}{dt} = \Phi = C\frac{d\theta_1}{dt}$$

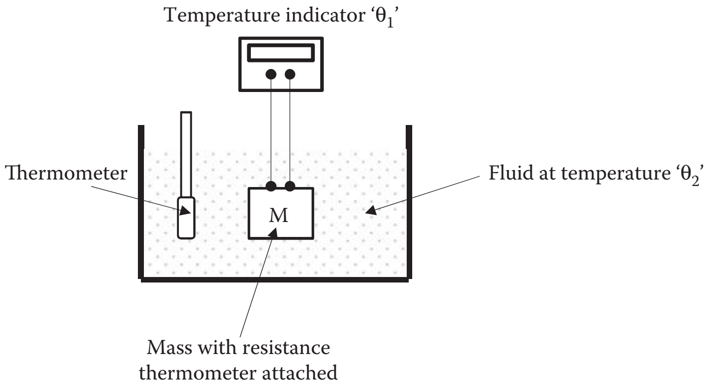


FIGURE 7.19 Diagram for an immersed mass.

The rate of heat transfer into the mass is $\Phi = C\,(d\theta_1/dt)$ and the rate of transfer is governed by the thermal resistance between the liquid and the mass. Ohm's law is similar so that

$$\Phi = \frac{(\theta_2 - \theta_1)}{R}$$

where R is the thermal resistance in kelvin/watt.

Equating for Φ

$$C\frac{d\theta_1}{dt} = \frac{\theta_1 - \theta_2}{R}$$

$$\frac{d\theta_1}{dt} = \frac{\theta_1 - \theta_2}{RC}$$

$$\frac{d\theta_1}{dt} + \frac{\theta_1}{RC} = \frac{\theta_2}{RC}$$

In mechanical, electrical, thermal and fluid systems, the product of resistance and capacitance is time constant ‘T’ therefore

$$\frac{d\theta_1}{dt} + \frac{\theta_1}{T} = \frac{\theta_2}{T}$$

Converting from a function of time to a function of ‘s’

$$s\theta_1 + \frac{\theta_1}{T} = \frac{\theta_2}{T}$$

$$\theta_1\,(Ts + 1) = \theta_2$$

$$\frac{\theta_1}{\theta_2}(s) = \frac{1}{(Ts + 1)}$$

Figure 7.20 depicts the block diagram for this transfer function.

Note: The transfer function is the same standard first-order equations that were derived for the spring–damper system and the thermal capacitance 'C' will be equivalent to 1/k and resistance will be equivalent to k_d.

7.2.6.2 Process Heating System

In the chemical process industry, precise control of heating being applied to process tanks is essential, particularly when dealing with volatile solutions such as oil where if overheating occurs would lead to a catastrophic event. In dangerous situations, pneumatics are used in preference to electronic controllers to reduce the risk of any fire that may occur.

Figure 7.21 shows a schematic diagram of an industrial heater using pneumatics for the control of the temperature of a tank of liquid that is heated by steam. The tank is fitted with a temperature sensor that sends a signal (within a range of 0.2–1.0 bar) to a temperature controller. The controller has an input temperature set by adjusting the control. A pressure sensing device produces another pneumatic signal between 0.2 and 1.0 bar depending upon the error that exists between the heating

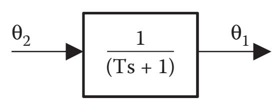


FIGURE 7.20 Block diagram for an immersed mass transfer function.

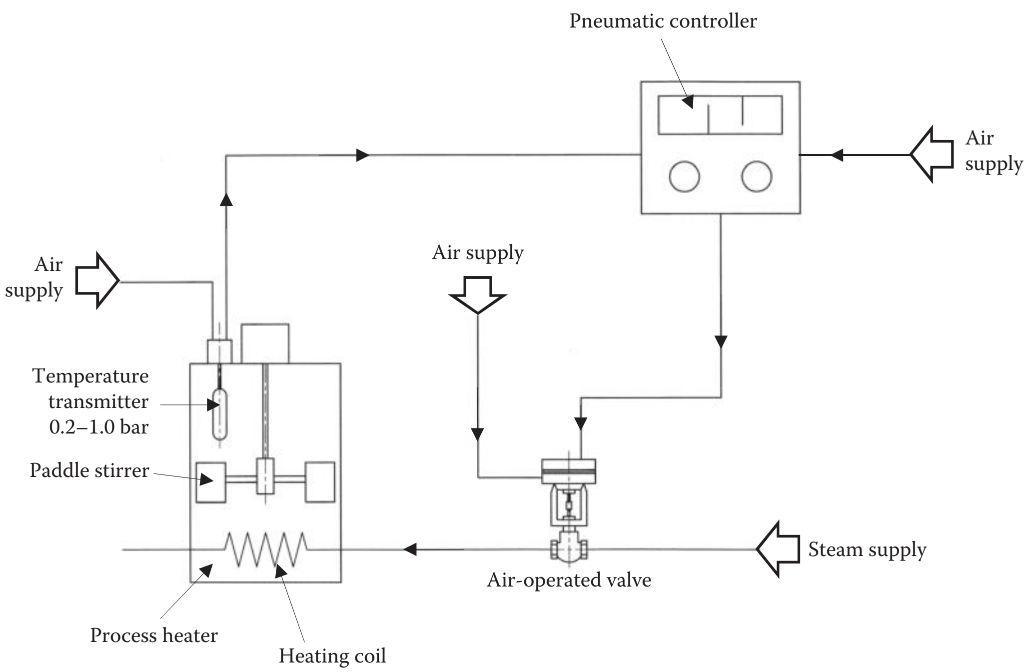


FIGURE 7.21 Typical pneumatic process heater control circuit.

tank sensor and the set temperature on the controller. This error signal is then sent to a pneumatically controlled flow control valve and this controls the flow of steam being sent to the heating coils within the tank. Generally, these tanks are fitted with stirrers to ensure that the temperature of the heated fluid is consistent throughout the tank. Hence, if the temperature sensor within the tank senses that the fluid is cooling, then additional steam is introduced to the heating coils and when any overheating is detected, the steam flow is reduced.

The model for this system will not be derived here as it is more complicated than simple:

$$\frac{\theta_1}{\theta_2}(s) = \frac{1}{(Ts + 1)}$$

Generally, these type of controllers are known as three term control (proportional, integration and differentiation) and will be covered later in the chapter.

EXAMPLE 7.3

Consider a simple thermal heating system that has a transfer function θ_1/θ_2 (s) = 1/(Ts + 1).

The temperature of the system is θ_o and this is at ambient temperature (20°C). When the set temperature is changed from 20°C to 100°C, the time constant 'T' is 4 s. Deduce a formula that shows how the system temperature changes with time.

SOLUTION

$$\frac{\theta_o}{\theta_i} = \frac{1}{T(s) + 1} \quad \therefore \; \theta_i = Ts\theta_o + \theta_o$$

Let θ_i be a constant (100°C) at all values of time after t = 0 (the start of the change).

$$\theta_i - \theta_o = Ts\theta_o$$

$$\theta_i - \theta_o = T\frac{d\theta_o}{dt}$$

Let $\theta_i - \theta_o = x$. Differentiating and $-d\theta_o = dx$, the equation will become

$$x = T\frac{d\theta_o}{dt}$$
$$= -T\frac{dx}{dt}$$

Integrating (without limits):

$$\theta_o = 100 - 80e^{-t/4} - \frac{t}{T} = \ln(x) + A$$

Substituting for 'x':

$$-\frac{t}{T} = \ln(\theta_i - \theta_o) + A$$

Now, when t = 0 (θ_o is the starting temperature and is equal to θ_i),

$$-\frac{t}{T} = 0 = \ln(\theta_i - \theta_i) + A$$

$$A = -\ln(\theta_i - \theta_i) \quad (\theta_i - \theta_i = \text{the change in temperature } \Delta\theta).$$

Substituting for 'A':

$$-\frac{t}{T} = \ln(\theta_i - \theta_o) - \ln(\Delta\theta)$$

$$= \ln\frac{(\theta_i - \theta_o)}{\Delta\theta}$$

Taking anti-logs:

$$e^{-t/T} = \frac{(\theta_i - \theta_o)}{\Delta\theta}$$

$$\Delta\theta e^{-t/T} = (\theta_i - \theta_o)$$

$$\theta_o = \theta_i - \Delta\theta e^{-t/T}$$

Substituting the value T = 4:

$$\Delta\theta = 100 - 20 = 80$$

Therefore, $\theta_i = 100$.
Hence, the law is

$$\theta_o = 100 - 80e^{-t/4}$$

Figure 7.22 evaluates and shows the resultant plot.

7.2.7 Hydraulic System

7.2.7.1 Hydraulic Motor

Figure 7.23 is a representation of a hydraulic motor in which an inlet is supplied with pressurised fluid at a flow rate Q m^3/s, at a pressure of p N/m^2. This will result in an output speed ω rad/s at a torque T Nm. It is assumed that the motor is 100% efficient and this will result in the outlet pressure being zero.

The following derivation of a model is for use in control theory. The formula relates flow rate 'Q' and the speed of rotation 'ω'

$$Q = k_q\omega$$

$$= k_q\frac{d\theta}{dt}$$

t s	$\theta_o C$
0	20.000
2	51.478
4	70.570
6	82.150
8	89.173
10	93.433
12	96.017
14	97.584
16	98.533
18	99.111
20	99.461

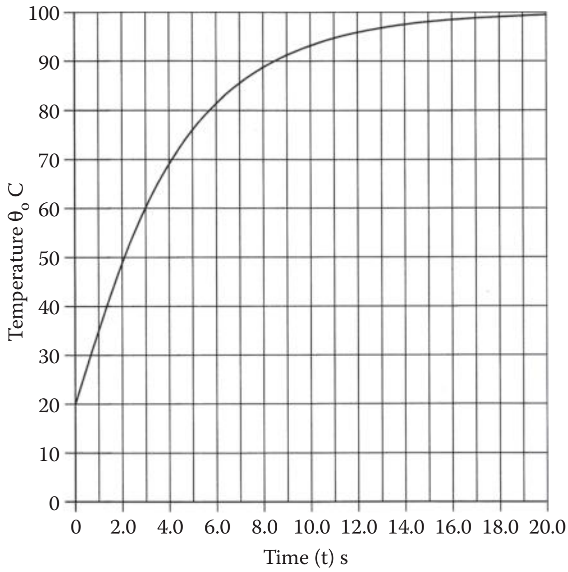


FIGURE 7.22 Resultant plot for Example 7.3.

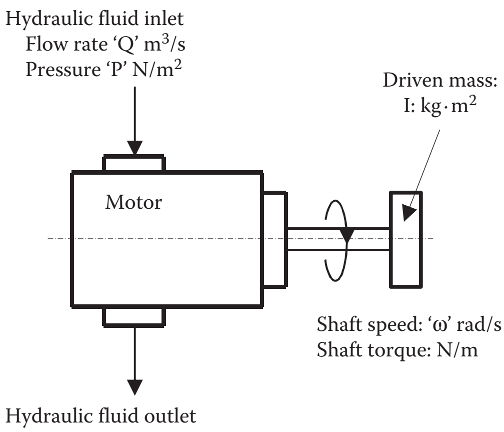


FIGURE 7.23 Hydraulic motor.

k_q is a constant known as the nominal displacement and has units of m^3 per radian. θ is the angle of rotation specified in radians.

Written as a function of 's', this is expressed as

$$Q = kq\ s\theta$$

If the flow rate is taken as the input and the angle of rotation is taken as the output, the transfer function will be

$$G(s) = \frac{\theta}{Q}$$

$$= \frac{1}{k_q s}$$

The following formula relates the system pressure 'p' to the output torque 'T'.

$$T = k_q \cdot p$$

The input pressure and output torque can be related as

$$G(s) = \frac{T}{p} = k_q$$

This is a further definition of the constant k_q.

EXAMPLE 7.4

If a hydraulic motor has a nominal displacement of 10 cm^3/radian, calculate the torque the motor will produce at a pressure of 110 bar.

Solution

$$\begin{aligned} T &= p \cdot k_q \\ &= 110 \times 10^5\ (N/m^2) \times 10 \times 10^{-6}\ (m^3/rad) \\ &= 110\ N \cdot m \end{aligned}$$

7.2.7.2 Hydraulic Cylinder

Where the hydraulic motor converts pressure into rotary motion, the hydraulic cylinder converts pressure into a linear motion. Figure 7.24a shows the basic elements of a double-acting cylinder.

A single-acting cylinder is similar but hydraulic pressure is only applied to one side of the cylinder and a spring returns the piston to its original position as shown in Figure 7.24b. In some instances, the spring may be replaced by a chamber of air.

The following discussion will be restricted to the double-acting cylinder.

The flow rate and piston movement are related by the law:

$$Q = A\frac{dx}{dt}$$

This can be expressed as a transfer function with 'x' as the output and 'Q' as the input.

$$\begin{aligned} G(s) &= \frac{x}{Q} \\ &= \frac{1}{As} \end{aligned}$$

Pressure and force are related by the law

$$F = \frac{p}{A}$$

and the corresponding transfer function with 'p' as the input and 'F' as the output will be

$$G(s) = \frac{x_o}{x_i} = \frac{1}{T^2s^2 + 2\delta Ts} \quad G(s) = \frac{F}{p} = A$$

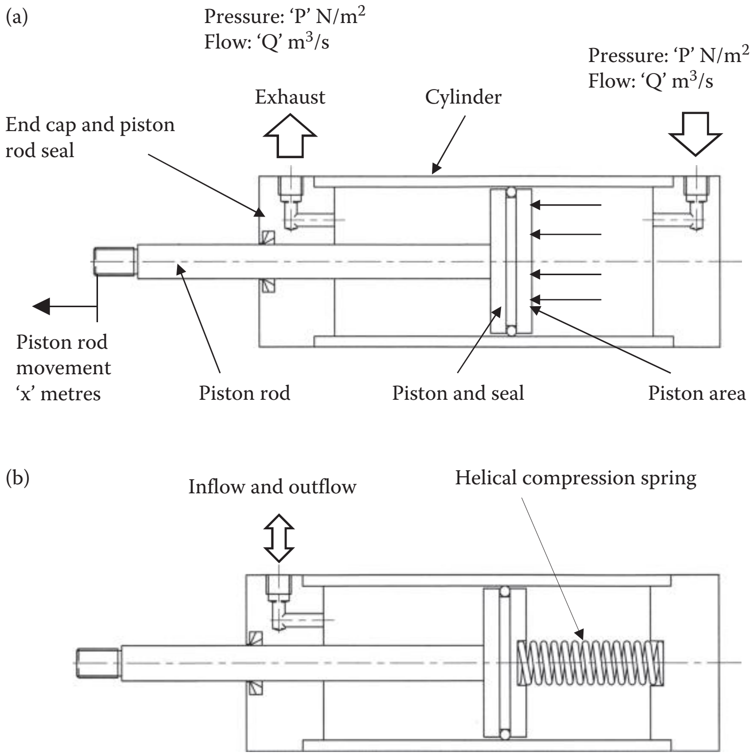


FIGURE 7.24 (a) and (b): Elements of double- and single-acting cylinders.

7.2.7.3 Directional Valve and Actuator

For a hydraulic cylinder (also referred to as an actuator) to function, a special type of valve, known as a directional control valve, is required. This controls the fluid flow from one side of the piston to the other.

Figure 7.25 represents such a valve connected to a double-acting cylinder.

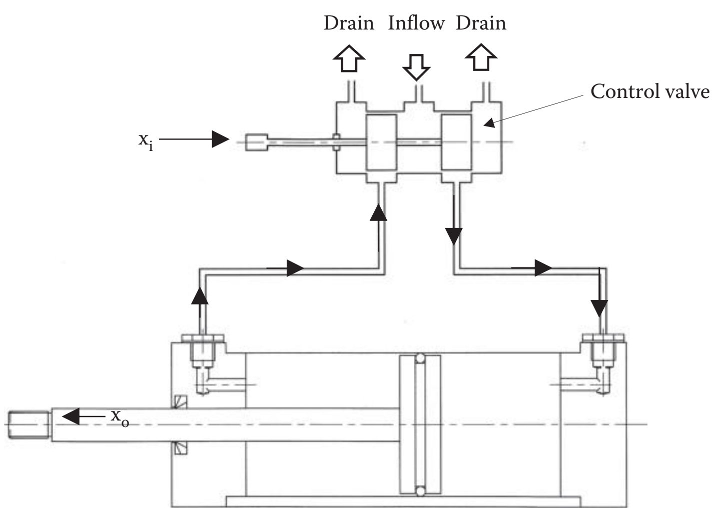


FIGURE 7.25 Directional control valve and actuator.

The input to the system is the x_i and this allows a flow of fluid into the cylinder of Q m^3/s; this will force the piston rod to move a distance 'x_o'.

If an assumption is made that for a constant supply pressure the flow rate is directly proportional to the valve position, it can be stated that:

$$Q = k_v \cdot x_i$$

k_v is the valve constant and the units are m^2/s.

Considering Figures 7.24a and 7.25, the area of the piston is A m^2.

The velocity of the piston is

$$v = \frac{dx_o}{dt}$$

This is related to the fluid flow and the piston area by the laws of continuity such that:

$$Q = k_v \cdot x_i$$

$$= A\frac{dx_o}{dt}$$

Changing this to a function of 's', this will become

$$k_v\, x_i = As \cdot x_o$$

Expressed as a transfer function

$$G(s) = \frac{x_o}{x_i}(s)$$

$$= \frac{1}{(A/k_v)s}$$

The units of A/k_v are seconds and it can be deduced that this is a time constant 'T'.

$$G(s) = \frac{x_o}{x_i}(s) = \frac{1}{Ts}$$

Note: This equation is not quite the standard first-order equation $1/(Ts + 1)$; the difference in the output is variable for a given input, unlike the previous examples where a limit is imposed on the output.

If the cylinder was replaced by a hydraulic motor, the equation would be similar where the output would be measured as an angle instead of linear motion.

EXAMPLE 7.5

A hydraulic cylinder has a bore of 100 mm and is controlled using a directional valve with a constant of 0.3 m^2/s.

1. Estimate the time constant 'T'.
2. Given that x_i and x_o are zero when $t = 0$, calculate the velocity of the piston and its output position after 0.1 s when the input is changed to 5.0 mm.

Solution

Area of the piston:

$$A = \frac{\pi D^2}{4}$$
$$= 7.854 \times 10^{-3}\ \text{m}^2$$

$$T = \frac{A}{k_v}$$
$$= \frac{7.854 \times 10^{-3}}{0.3}$$
$$= 0.026\ \text{s}$$

$$G(s) = \frac{x_o}{x_i}(s) = \frac{1}{Ts}$$

now

$$Ts \cdot x_o = x_i$$

$$T\frac{dx_o}{dt} = x_i$$

$$\frac{dx_o}{dt} = \text{velocity}$$

$$\text{velocity} = \frac{x_i}{T}$$

$$\text{velocity} = \frac{0.005\ \text{m}}{0.026}$$
$$= 0.191\ \text{m/s}$$

$$\text{velocity} = \frac{\text{distance}}{\text{time}}$$

$$\text{distance} = x_o = v \cdot t$$
$$= 0.191 \times 0.1$$
$$= 0.0191\ \text{m}$$

7.2.7.4 Directional Control Valve and Actuator Connected to a Mass

Considering that the previous example with a directional control valve is connected to the input and output of a double-acting actuator, this next example discusses when a mass is connected to the actuator and the mass is subject to a damping force.

Figure 7.26 shows the arrangement with the actuator subject to a hydraulic pressure. The applied force is due to the pressure F_d and is determined by the pressure acting on the area 'A' such that:

$$F_p = pA$$

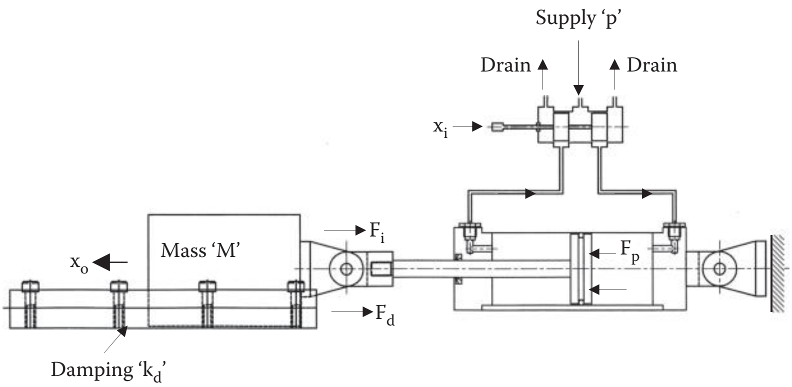


FIGURE 7.26 Double-acting cylinder attached to a mass.

The applied force is opposed by the inertia force F_i and the damping force F_d.

$$F_i = M\frac{d^2x_o}{dt^2}$$

and

$$F_d = k_d\frac{dx_o}{dt}$$

Balancing these forces will give

$$pA = M\frac{d^2x_o}{dt^2} + k_d\frac{dx_o}{dt}$$

Substituting $p = k_v \cdot x_i$

$$k_v \cdot x_i \cdot A = M\frac{d^2x_o}{dt^2} + k_d\frac{dx_o}{dt}$$

In Laplace form

$$k_v \cdot x_i \cdot A = Ms^2x_o + k_d s x_o$$

Rearranging into a transform function

$$G(s) = \frac{x_o}{x_i}(s) = \frac{1}{(M/Ak_v)s^2 + (k_d/Ak_v)s}$$

Examining the units, it will be found that

$$\frac{M}{Ak_v} = T^2 \quad \text{where T is a time constant}$$

The critical damping coefficient is

$$C_c = \sqrt{(4MAk_v)}$$

and the damping ratio is

$$\frac{\delta k_d}{C_c}$$

The transfer function then becomes

$$G(s) = \frac{x_o}{x_i}(s)$$

$$= \frac{1}{T^2s^2 + 2\delta Ts}$$

The block diagram for this transfer function is seen in Figure 7.27.

There is a similarity with the standard second-order equation $1/(T^2s^2 + 2\delta Ts + 1)$. The difference is due to there being no limitation on the output.

EXAMPLE 7.6

Consider that a double-acting hydraulic actuator with a bore of 100.0 mm and rod diameter of 25.0 mm moves a mass of 60 kg. It is fitted with a directional control valve with a constant of $k_v = 200{,}000$ Pa/m. There is a damping coefficient of 160 Ns/m.

Determine the time constant T, C_c and δ.
Given that x_i and x_o are zero when t = 0, calculate the initial acceleration of the mass when the output is changed suddenly to 6.0 mm.
Calculate the acceleration when the velocity reaches 2 mm/s.
Calculate the velocity when the acceleration is zero.

Solution

The piston area (A):

$$A = \frac{\pi(D^2 - d^2)}{4}$$

$$A = \frac{\pi(100.0^2 - 25.0^2)}{4}$$

$$= 7.363 \times 10^{-3}\ m^2$$

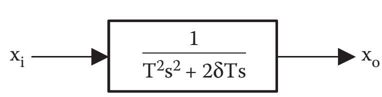


FIGURE 7.27 Block diagram for Figure 7.26.

$$T = \sqrt{\left(\frac{M}{Ak_v}\right)}$$

$$= \sqrt{\left(\frac{60}{20{,}000 \times 7.363 \times 10^{-3}}\right)}$$

$$= 0.638 \text{ s.}$$

$$C_c = \sqrt{(4\ MAk_v)}$$

$$= 187.997 \text{ N} \cdot \text{s/m}$$

$$\delta = \frac{k_d}{C_c}$$

$$\delta = 0.851$$

$$G(s) = \frac{x_o}{x_i} = \frac{1}{T^2s^2 + 2\delta Ts}$$

or in terms of time

$$x_i = (T^2 \times \text{acceleration}) + (2\delta T \times \text{velocity})$$

The initial velocity is zero.

$$0.005 = T^2\ a + 0$$

$$0.005 = 0.638^2\ a + 0$$

Therefore,

$$a = 6.939 \times 10^{-3} \text{ m/s}^2$$

Now when $v = 0.002$

$$0.005 = T^2\ a + (2\delta Tv)$$

$$0.005 = 0.638^2\ a + (2 \times 0.851 \times 0.638 \times 0.002)$$

$$a = 6.939 \times 10^{-3} \text{ m/s}^2$$

The system initially accelerates and will eventually settle down to a constant velocity with no acceleration.

Putting 'a' = 0

$$0.005 = 0 + (2 \times 0.851 \times 0.638) \times \text{velocity}$$

Hence,

$$\text{Velocity} = 0.004602 \text{ m/s} \quad \text{or} \quad 4.602 \text{ mm/s}$$

7.2.8 Electrical System Models

7.2.8.1 Resistance

When applying Ohm's law

$$V = I \cdot R \quad \text{or} \quad \frac{V}{I} = RG(s) = \frac{V}{I}(s) = sL \text{ or } s^2Q$$

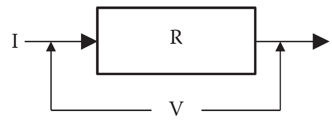


FIGURE 7.28 Schematic for a resistor.

This equation may be a function of time or ‘s’. The equation may be expressed in term of the electrical charge ‘Q’.

Now

$$\text{Since } I = \frac{dQ}{dt} \quad I(s) = sQ$$

Hence,

$$G(s) = \frac{V}{Q}(s) = sR$$

It will be seen that this expression is similar to that of the damper.

Figure 7.28 illustrates the symbol for a resistor.

7.2.8.2 Capacitance

The law for a capacitor is

$$Q = C \cdot V \quad \frac{V}{Q} = \frac{I}{C}$$

This expression is similar to that for a spring.

Differentiating with respect to time

$$\frac{dQ}{dt} = C\frac{dV}{dt}$$

where dQ/dt is the current ‘I’; hence, the equation can be expressed as

$$I = C\frac{dV}{dt}$$

and expressed as a function of ‘s’; this then becomes

$$I(s) = CsV$$

The transfer function is

$$G(s) = \frac{V}{I}(s) = \frac{1}{sC}$$

Figure 7.29 illustrates the symbol for a capacitor.

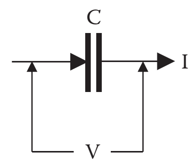


FIGURE 7.29 Schematic for a capacitor.

7.2.8.3 Inductance

From Faraday's law

$$V = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$$

This expression is similar to the model for mass and can be expressed as either a first-order or second-order equation.

As a function of 's'

$$V(s) = L\,sI \quad \text{or} \quad Ls^2Q$$

$$G(s) = \frac{V}{I}(s) = sL \quad \text{or} \quad s^2Q$$

Figure 7.30 is the circuit symbol for an inductor.

7.2.8.4 Potentiometer

Figure 7.31 illustrates two versions of a potentiometer:

- Figure 7.31a is a linear potentiometer.
- Figure 7.31b is an angular version.

In a potentiometer, if the supply voltage is constant and the current is very small or negligible, the output voltage 'V' is directly proportional to the position 'x' or angle 'θ'.

In this case, a simple transfer function is easily obtained.

$$V_o = V_i - \Delta V e^{-t/T} \quad G(s) = \frac{V_o}{x}(s) = \text{constant} = k_p \ \text{(linear)}$$

$$G(s) = \frac{V_o}{\theta}(s) = \text{constant} = k_p \ \text{(angular)}$$

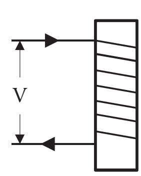


FIGURE 7.30 Schematic for an inductor.

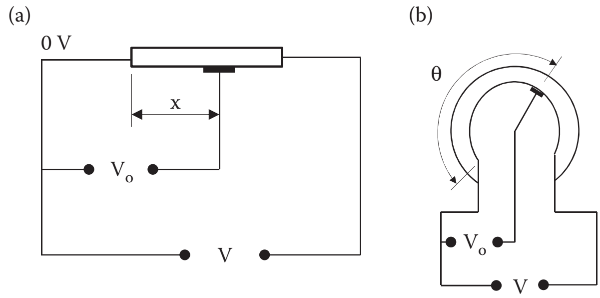


FIGURE 7.31 Schematics for linear (a) and angular (b) potentiometers.

7.2.8.5 R–C Series Circuit

Figure 7.32 illustrates a typical R–C series circuit in which the sum of the voltage across the resistor and the capacitor is

$$V_i = IR + \frac{I}{Cs}$$

$$V_i = I\left(R + \frac{1}{Cs}\right)$$

The output will be the voltage across the capacitor so

$$V_o = \frac{I}{Cs}$$

The transfer function is then

$$G(s) = \frac{V_o}{V_i}(s) = \frac{I/Cs}{I\left(R + 1/Cs\right)} = \frac{1}{RCs + 1}$$

The units of RC will be seconds and this is another electrical time constant 'T'. The transfer function can be written as

$$G(s) = \frac{V_o}{V_i}(s) = \frac{1}{RCs + 1}$$

This is a standard first-order equation and is the same as the spring and damper in the thermal example.

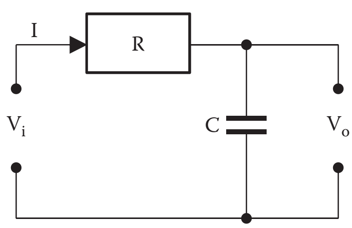


FIGURE 7.32 Schematic for an R–C circuit.

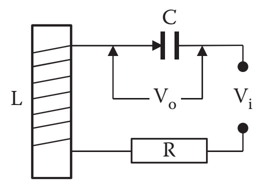


FIGURE 7.33 Schematic for L–C–R in series.

7.2.8.6 L–C–R in Series

This is three sub-systems in series. Figure 7.33 depicts such a circuit. In this case, the output is the voltage on the capacitor and the input is the voltage across the series circuit.

The input voltage is the sum of all three voltages and is found by summing them

$$V_i = IR + IsL + \frac{I}{sC}$$

The output voltage is $V_o = I/sC$. The transfer function is

$$G(s) = \frac{V_o}{V_i}(s) = \frac{I/Cs}{I(R + sL + 1/Cs)} = \frac{1}{RCs + CLs^2 + 1}$$

$$G(s) = \frac{V_o}{V_i}(s) = \frac{1}{s^2CL + sCR + 1}$$

If the units of CL are examined, it will be found to be seconds^2 and this is another time constant which is defined as $T^2 = CL$, the equation can then be rewritten as

$$G(s) = \frac{V_o}{V_i}(s) = \frac{1}{T^2s^2 + 2\delta Ts + 1}$$

The damping ratio δ is defined as

$$\delta + \frac{R}{\sqrt{4L/C}}$$

The expression $\sqrt{4(L/C)}$ is called the critical damping value.

Note: This is a standard second-order equation and is identical to the mass–spring–damper system.

EXAMPLE 7.7

A capacitance of 300 μF is connected in series with a resistor of 30 kΩ as shown in Figure 7.32.

The transfer function for this circuit is

$$\frac{\theta_o}{\theta_i}(s) = \frac{1}{(Ts + 1)}$$

There is a step voltage change from 3 to 10 V across the resistor.

t sec's	V_o
0	3.000
2	4.395
4	5.512
6	6.406
8	7.122
10	7.696
12	8.155
14	8.522
16	8.817
18	9.053
20	9.241
22	9.393
24	9.514
26	9.611
28	9.688
30	9.750

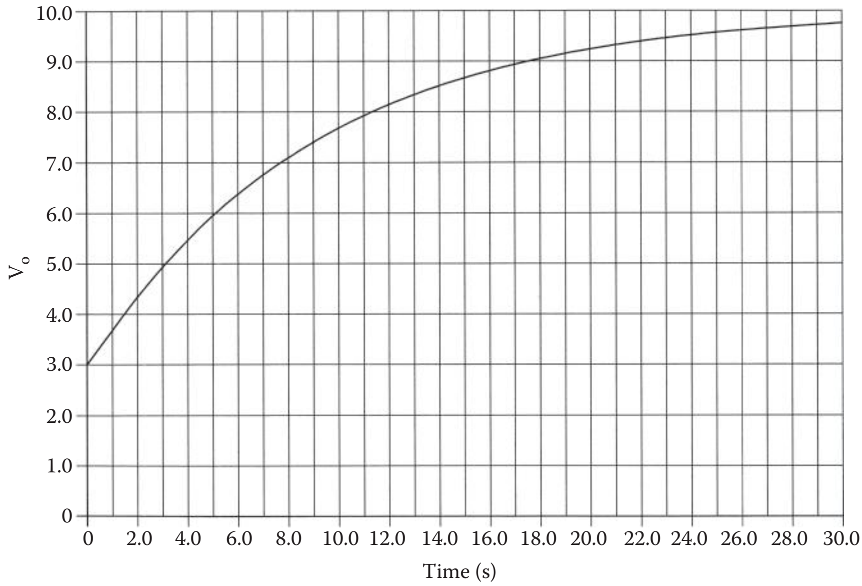


FIGURE 7.34 Results for Example 7.7.

Calculate the time constant 'T' and derive a formula for how the voltage across the capacitor varies with time.

SOLUTION

$$T = RC = 20 \times 10^3 \times 200 \times 10^{-6} = 9 \text{ s}$$

The derivation follows that used for Example 7.3 where 'V' replaces 'θ' and the following will result

$$V_o = V_i - \Delta V e^{-t/T}$$

Substituting the values T = 9

$$\Delta V = 10 - 3 = 7 \quad V_i = 10$$

$$V_o = 10 - 7e^{-t/9}$$

Figure 7.34 shows the results and corresponding plot for this equation. From the plot, it is clearly seen that the curve is converging to a value of $V_o = 10$ units.

7.2.9 Closed-Loop System Transfer Function with a Unity Feedback

Consider a simple system having an input θ_i and an output θ_o. For the system to be controlled where the output is to change and match the value of the input which will have a set value, the input will need to reflect the error θ_e instead of the set value. This error will be obtained by comparing the output value against the input value using a summing junction (see Section 7.3.3 for a full description). This will produce the result $\theta_e = \theta_i - \theta_o$ and as θ_o is subtracted from θ_i, this is referred to as a 'negative feedback'. The block diagram shows that the signal is returned around a closed loop to the summing junction, hence the name 'closed-loop system'.

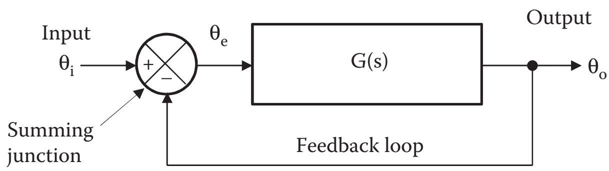


FIGURE 7.35 Closed-loop system with unity feedback.

Figure 7.35 shows the representative block diagram for such a system. This particular system is known as a 'unitary feedback system' as the feedback loop does not contain any processing element shown in the feedback loop (this will be discussed in more detail in Section 7.3.4).

$$G(s) = \frac{\theta_o}{\theta_e}$$

Substitute $\theta_e = \theta_i - \theta_o$.

$$G(s) = \frac{\theta_o}{\theta_i - \theta_o}$$

Divide the bottom line by θ_o.

$$G(s) = \frac{1}{(\theta_i / \theta_o) - 1}$$

Let

$$\frac{\theta_i}{\theta_o} - 1 = \frac{1}{G(s)}$$

$$\frac{\theta_i}{\theta_o} = \frac{1}{G(s)} + 1 = \frac{1 + G(s)}{G(s)} \quad \text{and invert}$$

$$\frac{\theta_o}{\theta_i} = \frac{G(s)}{1 + G(s)}$$

or

$$\frac{\theta_o}{\theta_i} = \frac{1}{1/(G(s) + 1)}$$

The transfer function for a closed-loop system with a unitary feedback is $\frac{\theta_o}{\theta_i} = \frac{1}{1/(G(s)+1)}$.

The transfer function for the open-loop system is G(s).

Considering the hydraulic valve and actuator as discussed in Section 7.2.7.3, when the actuator and valve are turned into a closed-loop system, the two transfer functions become

$G_{cl} = \frac{1}{Ts + 1}$ for the first-order version and $G_{cl} = \frac{1}{T^2 s^2} + 2\delta Ts + 1$ for the second-order version.

It will be seen that these two models are mathematically similar to the transfer function for the mass–spring–damper and L–C–R circuits.

EXAMPLE 7.8

An open-loop circuit has a transfer function $G(s) = 2/(s^2 + 2s + 1)$. Determine the closed-loop transfer function when the feedback is unity.

SOLUTION

$$G_{cl} = \frac{1}{1/(G(s) + 1)}$$

$$= \frac{1}{((s^2 + 2s + 1)/2) + 1}$$

$$= \frac{2}{s^2 + 2s + 1 + 2 + 1}$$

$$= \frac{2}{s^2 + 2s + 1 + 3}$$

7.3 BLOCK DIAGRAM AND TRANSFER FUNCTION MANIPULATIONS

In Section 7.2, block diagrams and transfer functions were introduced for various mechanical, hydraulic and electrical systems.

In this section, the reader will be introduced to 'open loop' and 'closed loop' control systems and the use of transfer functions in the study of these systems, but first a distinction needs to be made between these two types of systems.

To start, there are two general classifications of control systems:

1. Open-loop systems where the control action is independent of the output from the system.
2. Closed-loop systems in which the control action is dependent on the output.

7.3.1 Open-Loop Control System

An open-loop system requires an independent external action to ensure the required output. As an example, human intervention is needed when a saucepan is boiling on a heating ring to switch it off to prevent it boiling dry.

Open-loop systems have two important features:

1. They depend on their calibration for accurate operation.
2. Open-loop systems are not normally affected by stability problems; there is little risk that the input will result in an unexpected output.

Figure 7.36 gives an example of an open-loop system.

7.3.2 Closed-Loop Control System

Closed-loop systems are also referred to as 'feedback' systems where the output is compared to the input into the system. These types of systems are complicated, requiring the use of differential equations for their solution.

Figure 7.37 illustrates a typical closed-loop control system using a feedback loop together with its corresponding block as shown in Figure 7.38.

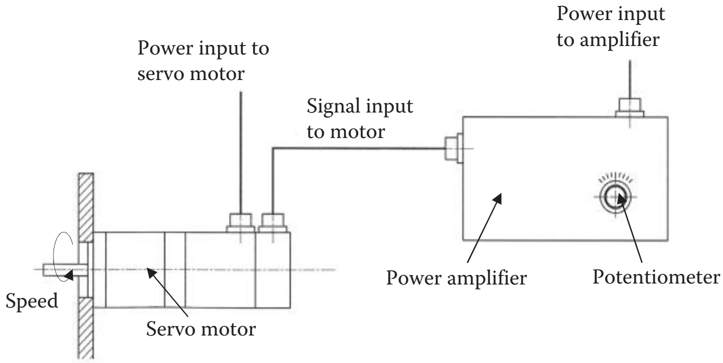


FIGURE 7.36 Open-loop control circuit.

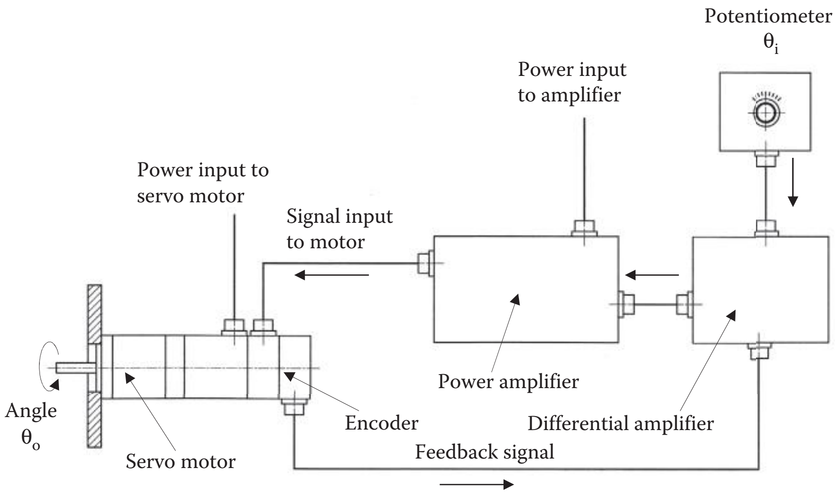


FIGURE 7.37 Closed-loop control circuit.

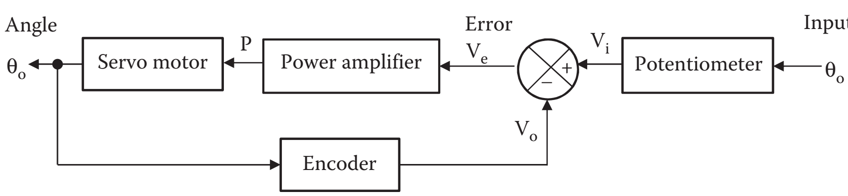


FIGURE 7.38 Bloch diagram for the closed-loop circuit in Figure 7.37.

7.3.3 Summing Junctions

In order for a control system to be regulated, the error between the output and input needs to be determined.

The summing junction is used to represent the addition or subtraction of signals.

Arrows are necessary to represent the direction of signal flow. Each incoming arrow has a sign associated with it to indicate whether the signal is positive or negative. The summing junction generally has two or more inputs and one output to which the sum of the inputs are routed as shown in Figure 7.39a and b. Figure 7.39a depicts a junction adding the two inputs together for a positive

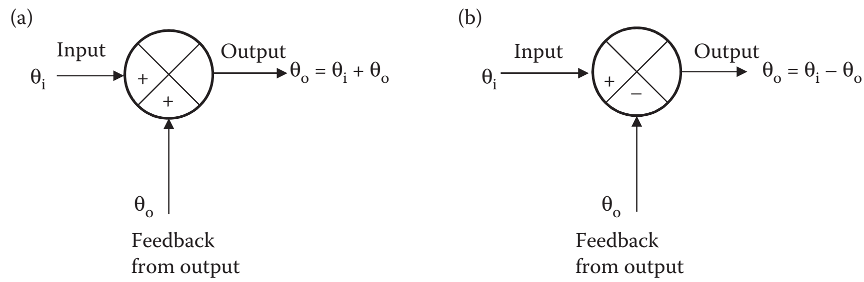


FIGURE 7.39 Summing junctions for (a) positive and (b) negative feedback.

output and Figure 7.39b subtracting the two inputs. In general, the junction shown in Figure 7.39b is the most common junction used in negative feedback systems. Very occasionally, the junction shown in Figure 7.39a may be found in positive feedback systems, but these will not be discussed in this chapter as they are specialised control techniques.

7.3.4 Closed-Loop System Transfer Functions

The basic block diagram for a closed-loop system with unitary feedback is shown in Figure 7.40. The main block is an *open-loop* system with a transfer function G_{ol}. This relates the output and the error so that $G_{ol} = \theta_o/\theta_i$.

The transfer function for a closed-loop system is G_{cl}. The input θ_i is related to the output θ_o.

By comparing the output value with the input value, the error is obtained in the summing junction. This will produce the result $\theta_e = \theta_i - \theta_o$ and as θ_o, is subtracted, this is referred to as 'negative feedback'. The block diagram shows that the signal is passed around a closed loop and this is known as a 'closed loop' system.

The system shown in Figure 7.40 is said to have a 'unitary feedback' as there is no processing shown in the feedback path.

The following is an important result for all closed-loop systems and will be used later.

$$G(s) = \frac{k}{(T^2s^2 + 2\delta Ts)}$$

$$\theta_e = \theta_i - \theta_o$$

$$G_{ol} = \frac{\theta_o}{\theta_i - \theta_o}$$

$$G_{ol}(\theta_i - \theta_o) = \theta_o$$

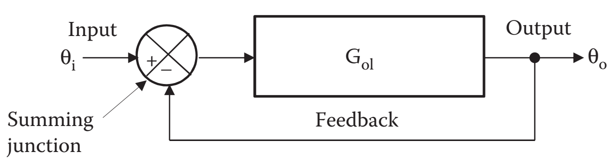


FIGURE 7.40 Basic block diagram for a closed loop with unity feedback.

$$G_{ol}\,\theta_i - G_{ol}\,\theta_o = \theta_o$$

$$G_{ol}\,\theta_i = \theta_o + G_{ol}\,\theta_o$$

$$G_{ol}\,\theta_i = \theta_o\,(1 + G_{ol})$$

$$G_{cl} = \frac{\theta_o}{\theta_i}$$

$$G_{cl} = \frac{G_{ol}}{1 + G_{ol}}$$

$$G_{cl} = \frac{1}{(1/G_{ol}) + 1}$$

This is the transfer function for a closed-loop unity feedback system, although in practice there will be a transducer to measure the output; therefore, a block in the feedback loop should be shown to represent the transducer as shown in Figure 7.41. There may be other signal processing instrumentation in the path such as amplifiers or signal conditioners and these will also need to be shown. In the figure, G_1 has been substituted in place of G_{ol}; this is the open-loop transfer function and is in the forward path and G_2 is in the feedback path.

The open-loop transfer function can be related to G_1 and G_2 as follows:

$$G = \frac{\theta_o}{\theta_e}$$

$$\theta_e = \theta_i - G_2\,\theta_o$$

$$G_1 = \frac{\theta_o}{\theta_i - G_2\theta_o}$$

$$G_1\,(\theta_i - G_2\,\theta_o) = \theta_o$$

$$G_1\,\theta_i - G_1\,G_2\,\theta_o = \theta_o$$

$$G_1\,\theta_i = G_1\,G_2\,\theta_o + \theta_o$$

$$G_1\,\theta_i = \theta_o\,(1 + G_1\,G_2)$$

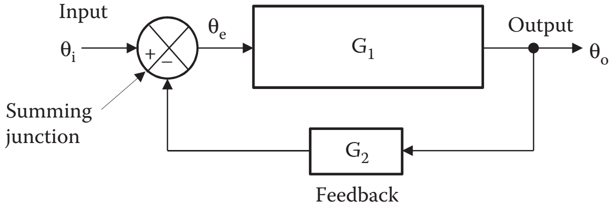


FIGURE 7.41 Basic block diagram for a closed loop with processor in the feedback loop.

$$G = \frac{\theta_o}{\theta_i} = \frac{G_1}{1 + G_1 G_2}$$

This is the transfer function for the closed system.

7.3.5 Velocity Feedback

When a system is unstable and tends to oscillate, a technique used to stabilise the system is to use velocity feedback. If the output is a motion x_o, the rate of change dx_o/d_t will be a true velocity.

A typical servo system uses both position and velocity feedback as shown in Figure 7.42. The velocity signal is generated from a tacho-generator or another suitable speed measuring device. The resulting signal is compared with the input and output and the resulting error signal will be

$$x_e = x_i - k_2 x_o - k_1 \frac{dx_o}{dt}$$

When a step change is made to the input, the error will be a maximum so the output will change very rapidly. The velocity feedback will be greatest at the start. The effect of the feedback will be to reduce this error directly proportionally to the velocity. When the output is static, the feedback will be zero and therefore no error will result. The feedback will have the same effect as damping and if an analysis is completed, it will be seen that there will be control over k_1 giving control over damping. This method is very useful for stabilising an oscillating system.

EXAMPLE 7.9

Figure 7.43 depicts a closed-loop control system having both velocity and negative feedback. The transfer functions for the system is $G(s) = k/(T^2 s^2 + 2\delta Ts)$. It is required to derive the closed-loop transfer function.

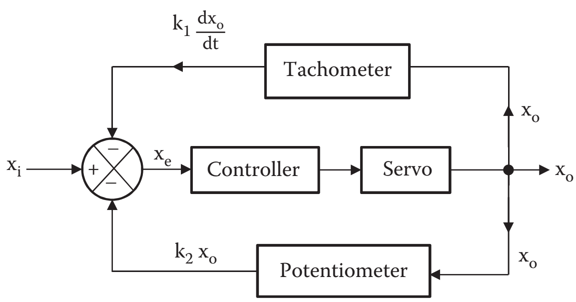


FIGURE 7.42 Block diagram for a closed-loop system with velocity feedback.

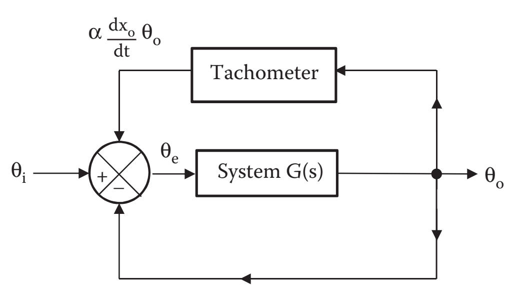


FIGURE 7.43 Closed-loop circuit for Example 7.8.

Solution

The open-loop transfer function is given as

$$G(s) = \frac{k}{(T^2s^2 + 2\delta Ts)}$$

$$\frac{\theta_o}{\theta_i} = \frac{k}{T^2s^2 + s(2\delta T + k\alpha) + k}$$

$$\theta_e = \theta_i - \theta_o - (\alpha s + 1)$$

$$\theta_o = G\theta_e = G[\theta_i - \theta_o(\alpha s + 1)]$$

$$\theta_o = G\theta_i - G\theta_o(\alpha s + 1)$$

$$\theta_o + G\theta_o(\alpha s + 1) = G\theta_i$$

$$\theta_o[1 + G(\alpha s + 1)] = G\theta_i$$

$$\frac{\theta_o}{\theta_i} = \frac{G}{1 + G(\alpha s + 1)}$$

$$\frac{\theta_o}{\theta_i} = \frac{1}{1/G + (\alpha s + 1)}$$

Substituting $G = \dfrac{k}{(T^2s^2 + 2\delta Ts)}$

$$\frac{\theta_o}{\theta_i} = \frac{1}{[T^2s^2 + 2\delta Ts]/k + (\alpha s + 1)}$$

$$\frac{\theta_o}{\theta_i} = \frac{k}{T^2s^2 + 2\delta Ts + k(\alpha s + 1)}$$

$$\frac{\theta_o}{\theta_i} = \frac{k}{T^2s^2 + s(2\delta T + k\alpha) + k}$$

This is the closed-loop transfer function.
The term with 's' is the effective damping term and is affected by the value of $k\alpha$.

7.3.6 Disturbance

Figure 7.44 characterises the effect on the output signal by a disturbance.
The disturbance 'd' is added to the output 'θ' to produce a new output θ_o.
G is the forward path transfer function

$$\theta_e = \theta_i - \theta_o$$

$$\theta_o = \theta + d$$

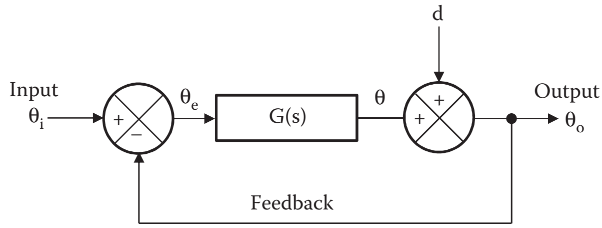


FIGURE 7.44 Closed-loop circuit subject to a disturbance.

$$\theta = G\theta_e$$

$$\theta_o = G\theta_e$$

$$\theta_o = G\theta_e + d$$

$$\theta_o = G(\theta_i - \theta_o) + d$$

$$\theta_o = G\theta_i - \theta_o + d$$

$$\theta_o(1 + G) = G\theta_i + d$$

$$\theta_o = \frac{(G\theta_i + d)}{(1 + G)}$$

EXAMPLE 7.10

A simple closed-loop circuit as shown in Figure 7.44 consists of an amplifier with a gain of 10. For an input of 4 mA, show the effect of a disturbance when added to the output with a magnitude of

1. 0
2. 2

Solution

1. Let $G = 10$, $\theta_i = 4$, $d = 0$.

$$\theta_o = \frac{(10 \times 4 + 0)}{(1 + 10)}$$

$$\theta_o = \frac{40}{11}$$

2. Let $G = 10$, $\theta_i = 4$, $d = 2$

$$\theta_o = \frac{(G\theta_i + d)}{(1 + G)}$$

$$\theta_o = \frac{(10 \times 4 + 2)}{(1 + 10)}$$

$$\theta_o = \frac{42}{11}$$

This result shows that a disturbance of 2 will produce an output error of 2/11.

7.3.6.1 To Eliminate the Effect of a Disturbance

To reduce or eliminate the effect of a disturbance acting on the output, it is possible to introduce a special feedback path. Figure 7.45 shows the idealised system.

The disturbance 'D' in this case is processed through a transfer function 'G_2' and this is added to the input. With G_1 as the forward path transfer function, G_2 is the feedback path transfer function.

$$\theta = G_1\,\theta_e = G_1\,(\theta_i - G_2\,D) \quad \theta_o = \theta + D$$

$$\theta_o = G_1\,(\theta_i - G_2\,D) + D \quad \theta_o = G_1\,\theta_i - G_1\,G_2\,D + D$$

From the last line, it will be seen that if $G_1G_2D = D$, then $\theta = G_1\theta_i$. The effect of the disturbance will be completely removed when $G_1 = 1/G_2$.

EXAMPLE 7.11

For the system described in Section 7.3.6.1, the forward path transfer function is

$$G = \frac{12}{4s^3 + 7s^2 + 60s^2} = \frac{12}{4s^3 + 60s^2} \quad G_1(s) = \frac{4}{(s+1)}$$

Determine the transfer function for the feedback path to eliminate the effects of the disturbance acting on the input.

Solution

$$G_2(s) = \frac{1}{G_1(s)} = \frac{(s+1)}{4}$$

7.3.7 Proportional and Differential Control

A transfer function used in process engineering is the proportional plus differentiation (P + D) transfer function. Figure 7.46 depicts the block diagram for this function.

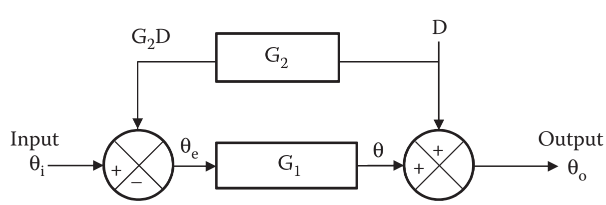


FIGURE 7.45 Closed-loop circuit elimination of disturbance.

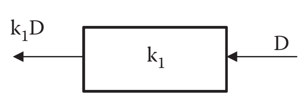


FIGURE 7.46 Feedback diagram for proportional control.

For a proportional system, the output is directly proportional to the input; it can be considered as an amplifier or attenuator with k_1 as a simple ratio.

Figure 7.47 shows a differential block in which the output is proportional to the rate of change of the input with time. Fitting a tachometer to the output is such an example. In Laplace form, the output is k_2sD.

The units of k_2 are in seconds and can be written as $k_2 = Tk_1$ where T is the time derivative.

Proportional control can be combined with differential control and Figure 7.48 illustrates such a block. For P + D, it follows that the transfer function is k_1 (Ts + 1).

EXAMPLE 7.12

Given that the open-path transfer function is $G_1 = 5/(s + 2)$, determine the value of the time derivative 'T' and the constant which will eliminate the disturbance 'D'.

Figure 7.49 illustrates the block diagram.

Solution

$$G_2 = \frac{1}{G_1} = \frac{(s+2)}{5}$$

$$G_2 = k_1\ (Ts + 1)$$

Rearranging so that the forms match

$$0.2(s + 2) = k_1(Ts + 1)$$

$$0.4(0.5\ s + 1) = k_1(Ts + 1)$$

Hence, by comparison, $k_1 = 0.4$ and $T = 0.5$.

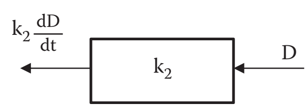


FIGURE 7.47 Feedback block diagram for differential control.

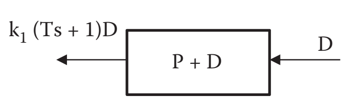


FIGURE 7.48 Feedback block diagram for proportional and differential control.

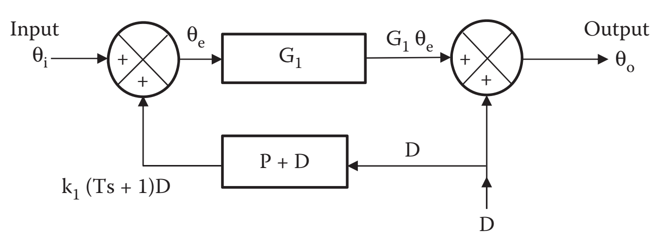


FIGURE 7.49 Closed-loop circuit for Example 7.11.

7.3.8 Simplifying Complex Systems

The symbol 'H' is commonly used for the feedback transfer function but any appropriate letter can be used to help simplify complex circuits.

Figures 7.50 through 7.52 show the stages in simplifying a block diagram to a single block with one transfer function.

EXAMPLE 7.13

Consider the block diagram shown in Figure 7.53a and derive the transfer function with

$$G_1 = \frac{3}{s}, \quad G_2 = \frac{1}{(4s+5)}, \quad G_3 = 4, \quad G_4 = \frac{1}{s}, \quad H_1 = 5, \; H_2 = 0.5$$

Solution

Figure 7.53c shows the simplification of the block diagram that is shown in Figure 7.53a to a single block and transfer function.

Substituting the above data results in

$$D_1 = \frac{4/(4s+5)}{1 + [(4 \times 0.5)/(4s+5)]} = \frac{4}{(4s+5)[1 + (2/(4s+5))]} = \frac{4}{4s+7}$$

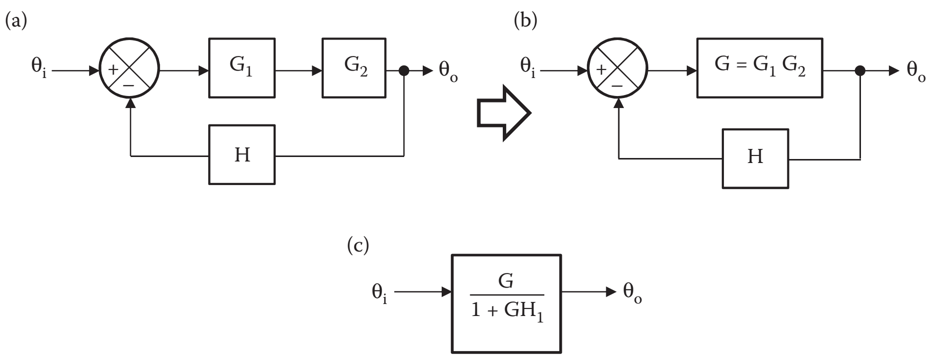


FIGURE 7.50 (a), (b) and (c): Reducing a block diagram, complexity (1).

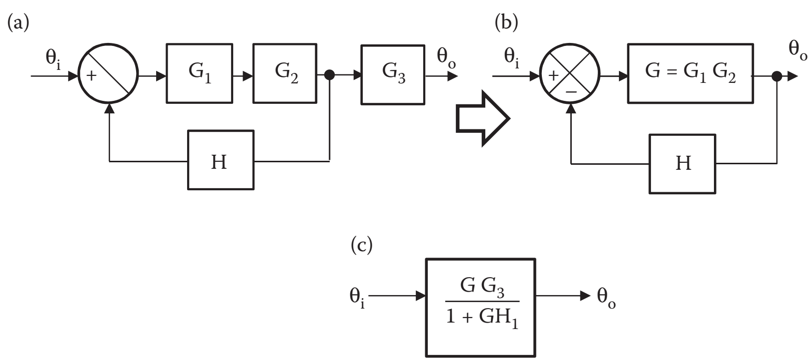


FIGURE 7.51 (a), (b) and (c): Reducing a block diagram, complexity (2).

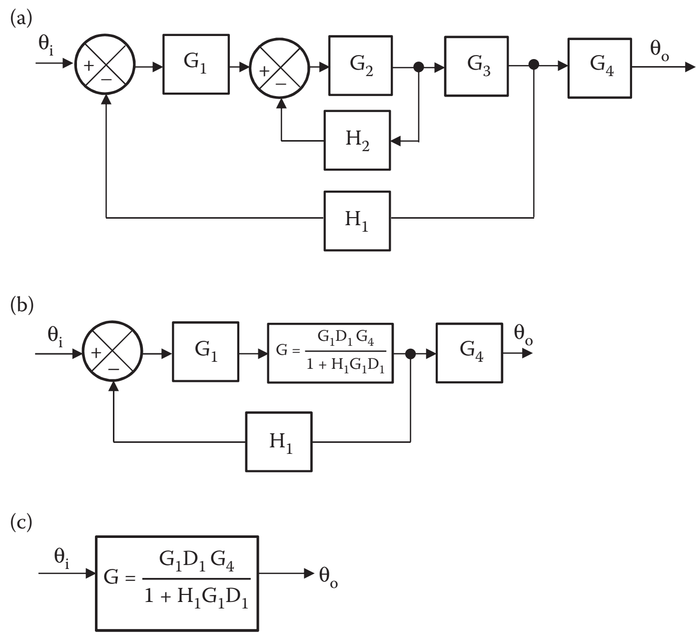


FIGURE 7.52 (a), (b) and (c): Reducing a block diagram, complexity (3).

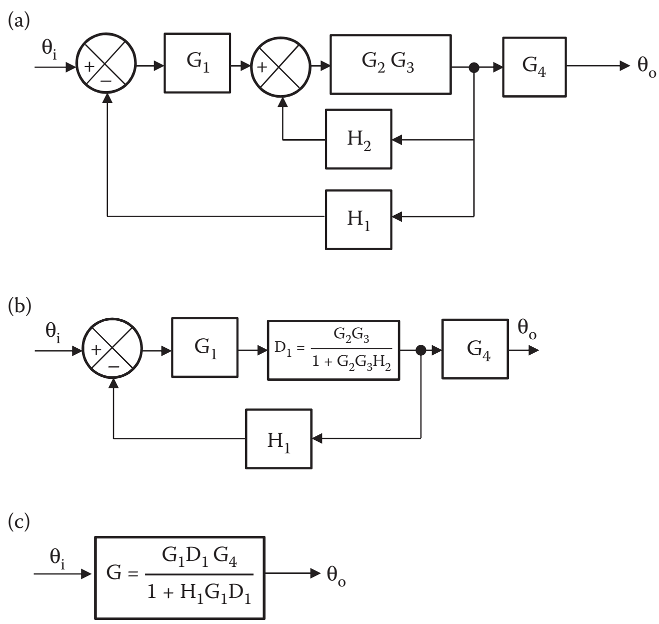


FIGURE 7.53 (a), (b) and (c): Reducing a block diagram for Example 7.13.

$$G = \frac{(3/s)[4/(4s+7)]D_1}{1+(3/s)(5)(D_1)} = \frac{(3/s^2)[4/(4s+7)]}{1+(15/s)[4/(4s+7)]} = \frac{12/[s^2(4s+7)]}{1+60/(4s^2+7s)}$$

$$= \frac{12}{s^2(4s+7)+s^2(4s+7)[60/(4s+7)]}$$

$$G = \frac{12}{4s^3+7s^2+60s^2} = \frac{12}{4s^3+67s^2}$$

8 Heat and Temperature

8.1 HEAT

Matter is made up of molecules that are continuously in a state of movement; the amount of movement is dependent on how much energy the matter possesses. As more energy is added, the molecules become more active, moving around very rapidly, and the hotter the matter becomes.

As energy is removed from the matter or body, the molecules become less active, their speed reduces and the matter then becomes colder.

Heat energy may be added to a body by a number of means including the direct application of a flame, electrical resistance, friction and so on.

8.1.1 Temperature

The heat of a body may be determined by measuring it using a thermometer. The most common thermometers used consist of a liquid (usually alcohol) within a glass tube that is closed at both ends and evacuated of air. These liquid thermometers are based on the principle of thermal expansion. As a substance is heated, it expands to a greater volume. Nearly all substances exhibit this behaviour of thermal expansion and are the basis of the design and operation of all liquid thermometers. There are other types of thermometers that give a digital readout of the temperature and these rely on the change of electrical resistance of a wire, usually platinum.

Another type of thermometer depends upon the electromagnetic force change with temperature of two distinct metals (copper and iron) that are joined together using solder.

The ends of the wires are connected to a galvanometer and as the junction is heated, a current flows within the circuit causing the galvanometer to deflect.

All thermometers have to be calibrated so that they will measure absolute temperatures and this is undertaken by measuring two distinct temperatures. First, the thermometer (or junction) is immersed in freezing liquid that is on the point of thawing and the level of the liquid is measured. The thermometer is then immersed in boiling water and the second measurement is then made. These measurements are made at an atmospheric pressure of 1 atm.

8.1.1.1 Temperature Scales

There are currently three main temperature scales used in engineering and science for measuring temperature:

Celsius: For thermometers measuring in the Celsius scale, the freezing water is marked at 0 and the boiling point of water is marked at 100. The scale is then divided into 100 equal divisions between the marks. Accurate measurements can then be made of the temperature of any object within the temperature range for which the thermometer has been calibrated.

Fahrenheit: In a similar manner to that described above, the freezing point of water is measured at 32°F and boiling point of water at 212°F. The scale is then divided into 180 divisions.

Kelvin: While the Celsius and Fahrenheit scales are widely used today, there are other scales that have been developed; the Rankine scale, the Newton scale and the Romer scale. All of these are very rarely used today. Yet another scale, the *Kelvin temperature* *scale,* has been adopted as the standard metric system of temperature measurement and is possibly the scale most widely used by engineers and scientists. The Kelvin scale is similar to the Celsius scale in that there are 100 divisions between the freezing and boiling points of water. However, the zero degree mark on the Kelvin scale is 273.15 units cooler than it is on the Celsius scale. Hence, the temperature of 0 Kelvin is equivalent to a temperature of –273.15°C. Note that the degree symbol is not used on the Kelvin scale, so that a temperature of 250 units above 0 Kelvin is referred to as 250 Kelvin and not as 250° Kelvin; the temperature is abbreviated as 250 K. Conversions between Celsius temperatures and Kelvin temperatures can be performed using one of the two following equations:

$$°C = K - 273.15° \tag{8.1}$$

$$K = °C + 273.15° \tag{8.2}$$

The zero point on the Kelvin scale is referred to as 'absolute zero' and it is the temperature where all movement of molecules making up matter becomes stationary. Currently, engineers and scientists have been able to lower the temperature to just above the absolute zero point but as yet not quite reached it.

8.1.1.1.1 Converting between Temperature Scales

Celsius scale: To convert from degrees Celsius to degrees Fahrenheit, the following equation is used:

$$°F = (1.8 \times °C) + 32 \tag{8.3}$$

EXAMPLE 8.1

Convert 20°C to the equivalent scale in Fahrenheit.

$$°F = (1.8 \times 20) + 32$$

$$= 68°F$$

Fahrenheit scale: Temperatures measured in the Fahrenheit scale can be converted to the equivalent Celsius scale using the following formula:

$$°C = \frac{(°F - 32)}{1.8} \tag{8.4}$$

EXAMPLE 8.2

Convert 68°F to the equivalent scale in Celsius.

$$°C = \frac{(68 - 32)}{1.8}$$

$$= 20°C$$

8.1.2 Thermal Expansion

Thermal expansion can affect solids, liquids and gases and there will be a change in dimension when there is a change in temperature. When there is an increase in temperature in a solid, there

TABLE 8.1
Thermal Expansion Coefficients for Various Materials

Material	Linear $\alpha(10^{-6}\ K^{-1})$	Material	Linear $\alpha(10^{-6}\ K^{-1})$
Aluminium	23.1	Silver	18.9
Brass	20.3	Solder (lead–tin)	25.0
Carbon (graphite)	6.5	Steel, stainless	17.3
Chromium	4.9	Steel, structural	12.0
Copper	16.5	Tin	22.0
Gold	14.2	Titanium	8.5
Invar (64% Fe, 36% Ni)	1.2	Tungsten	4.5
Iron	11.8	Uranium	13.9
Lead	28.9	Water, ice (0°C)	51.0
Nickel	13.3	Zinc	30.2

will be an increase in the length, thickness and height. When the substance is either a liquid or gas, it is more useful to describe the expansion in terms of a change in volume.

The thermal expansion is generally uniform in all dimensions; there will be different expansion coefficients and these are characteristic for elements and compounds. Table 8.1 tabulates the coefficients for thermal linear expansion for a range of materials.

For a solid, the changes in the characteristic dimensions are due to a change in temperature:

Linear expansion:

$$\Delta L = L_o \alpha \Delta T \tag{8.5}$$

Area:

$$\Delta A = A_o 2\alpha \Delta T \tag{8.6}$$

Volumetric:

$$\Delta L = L_o 3\alpha \Delta T \tag{8.7}$$

EXAMPLE 8.3

A steel bridge is built in several sections; each section is 20 m long. The expansion gap between each section is 30 mm at an ambient temperature of 18°C. Determine the maximum temperature that can withstand before buckling occurs.

SOLUTION

From Equation 8.5:

$$\Delta L = L_o \alpha \Delta T$$

where

$\alpha = 12.0^{-6}$ mm per °C
$T_1 = 18.0$°C
$\Delta L = 20{,}000$ mm $\times\ 12.0^{-6}$ mm per °C $(T^2 - 18.0)$

Hence,

$$T_2 = 18.0°C + \frac{30.0\text{ mm}}{(20{,}000\text{ mm} \times 12.0^{-6}\text{ mm/°C})}$$

$$T_2 = 143°C$$

The highest temperature on the earth was in Death Valley in California with a temperature of 56.7°C (134°F), which has a good temperature reserve factor.

8.1.3 Heat Capacity

The total heat capacity (C) of a body is the amount of heat energy required to raise the temperature of the body by 1 degree. It is expressed in Joules per degree K ($J \cdot K^{-1}$).

The specific heat capacity (c) of a substance is the heat required to raise a unit mass (m) through 1 degree; it is the heat capacity per unit mass of the substance and its units are expressed in either Joule per gramme per degree K ($J\ g^{-1}\ K^{-1}$) or Joule per kilogramme per degree K ($J\ kg^{-1}\ K^{-1}$). The specific heat of water (c_w) is approximately 4200 $J\ kg^{-1}\ K^{-1}$.

The kilocalorie is the amount of heat required to raise 1 kilogramme of water by 1 degree Celsius. In SI units, the calorie is equivalent to 4.184 J. These reversible conversions of work and heat energy are called the 'mechanical equivalent of heat'.

From the definition of specific heat capacity, it follows that:

$$\text{Heat capacity, } C = \text{mass} \times \text{specific heat capacity} \tag{8.8}$$

As an example, the specific heat capacity of copper is 0.39 $J\ g^{-1}\ K^{-1}$ or 390 $J\ kg^{-1}\ K^{-1}$.

$$\begin{aligned}\text{The heat capacity of 5 kg of copper} &= 5 \times 390 \\ &= 1950\ J\ K^{-1} \\ &= 1.95\ kJ\ K^{-1}\end{aligned}$$

Table 8.2 gives the specific heat capacities for a range of materials.

TABLE 8.2
Specific Heat Capacities for Various Materials

Material	Sp Ht: $J\ kg^{-1}\ K^{-1} \times 10^3$	Material	Sp Ht: $J\ kg^{-1}\ K^{-1} \times 10^3$
Aluminium	0.91	Solder	0.18
Brass	0.38	Glass	0.70
Copper	0.39	Ice	2.10
Iron	0.47	Rubber	1.70
Steel	0.45	Stone	0.90
Lead	0.13	Wood	1.70
Mercury	0.14	Alcohol	2.50
Nickel	0.46	Ether	2.40
Platinum	0.13	Paraffin oil	2.10
Silver	0.24	Turpentine	1.76

EXAMPLE 8.4

It is required to raise the temperature of 20 kg of steel from a temperature of 20°C to 250°C.
From Table 8.1, the specific heat of steel is 450 J/kg °C.
Estimate the amount of heat required.

Solution

From Equation 8.8

$$C = m \times c\ (T_2 - T_1)$$

where
C = total heat energy in Joules, (J)
m = mass of material (20 kg)
c = specific heat capacity in J/kg °C (450 J/kg °C)
$(T_2 - T_1)$ = temperature difference (250°C – 20°C)
C = 20 kg × 450 J/kg °C × (250 – 20)°C
= 2.07 MJ

8.1.4 Heat Transfer

Heat is transferred from one location to another by one of three methods:

- Conduction
- Convection
- Radiation

8.1.4.1 Conduction

Consider an iron bar that is partially placed in a furnace to heat an area of the bar for a blacksmith to work on.

The heat from the furnace will travel along the bar gradually before it becomes too hot to hold the bar comfortably.

The transfer rate (H) is the ratio of the amount of heat per time for the heat to transfer from one location to another; this may be expressed as

$$H = \frac{Q}{\Delta t} \tag{8.9}$$

where
H has units of J/s or Watts
Q is in Joules
t is in seconds

There has to be a temperature difference for conduction to take place as heat will only flow from a hot body to a cold body and not the other way around.

Considering that the iron bar has a length (L) and a cross-sectional area (A), the heat conduction along the length of the bar is given by

$$H = \frac{Q}{\Delta t} = kA\frac{(T_2 - T_1)}{L} \tag{8.10}$$

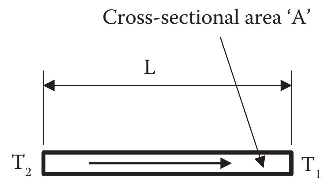


FIGURE 8.1 Heat conduction along an iron bar.

TABLE 8.3
Thermal Conductivities for a Range of Materials

Material	'k' ($W\ m^{-1}\ K^{-1}$)	Material	'k' ($W\ m^{-1}\ K^{-1}$)
Aluminium	210	Platinum	0.13
Brass	109	Silver	420
Copper	380	Solder	55 (approx.)
Iron (pure)	76	Glass	1.1
Iron (wrought)	59	Ice	2.10
Steel	46	Rubber	1.70
Lead	35	Wood	0.21
Mercury	9.2	Alcohol	0.18
Nickel	87	Paraffin oil	0.13

where heat flows from position T_2 to T_1 and $T_2 > T_1$ as shown in Figure 8.1.

The constant (k) is called 'thermal conductivity'. Table 8.3 gives values of thermal conductivities for a range of materials.

It will be noted that the metal materials have a far higher conductivity constant than liquids or other non-metal materials.

8.1.4.2 Convection

Liquids and gases are fluids. The particles in these fluids can move from place to place. Convection occurs when particles with a lot of heat energy in a liquid or gas move and take the place of particles with less heat energy. Heat energy is transferred from hot places to cooler places by convection.

Liquids and gases expand when they are heated. This is because the particles in liquids and gases move faster when they are heated than they do when they are cold. As a result, the particles take up more volume. This is because the gap between particles widens, while the particles themselves stay the same size.

The liquid or gas in hot areas is less dense than the liquid or gas in cold areas, so it rises into the cold areas. The denser cold liquid or gas falls into the warm areas. In this way, convection currents that transfer heat from place to place are set up.

8.1.4.3 Radiation

Thermal radiation is a process by which energy, in the form of electromagnetic radiation, is emitted by a heated surface in all directions and travels directly to its point of absorption at the speed of light; thermal radiation does not require an intervening medium to carry it.

Thermal radiation ranges in wavelength from the longest infrared rays through the visible-light spectrum to the shortest ultraviolet rays. The intensity and distribution of radiant energy within this range is governed by the temperature of the emitting surface. The total radiant heat energy emitted by a surface is proportional to the fourth power of its absolute temperature (the Stefan–Boltzmann law).

TABLE 8.4
Comparisons between Two Radiator Surfaces

Colour	Finish	Ability to Emit Thermal Radiation	Ability to Absorb Thermal Radiation
Dark	Dull or matt	Good	Good
Light	Shiny	Poor	Poor

$$E = \sigma T^4 \tag{8.11}$$

where

E is the radiant heat energy emitted from a unit area in one second.

σ is the constant of proportionality (the Stefan–Boltzmann constant).

T is the absolute temperature (Kelvin).

The constant has a value of $5.6704 \times 10^{-8}\ \mathrm{Wm}^{-2} \cdot \mathrm{K}^{-4}$.

The law applies only to blackbodies and theoretical surfaces that absorb all incident heat radiation. The rate at which a body radiates (or absorbs) thermal radiation depends upon the nature of the surface as well. Objects that are good emitters are also good absorbers (Kirchhoff's radiation law). A blackened surface is an excellent emitter as well as an excellent absorber. If the same surface is silvered, it becomes a poor emitter and a poor absorber. A blackbody shown in Table 8.4 gives a comparison between two radiant surfaces.

A fire in an open grate is an excellent example of thermal radiation as the heat is felt directly without any intervening fluid in between the radiant and the recipient.

A common misconception in home heating is the use of the word 'radiator'. This is a thin panel fed using hot water; radiators are often painted with white gloss paint. They would be better at radiating heat if they were painted with black matt paint, but in fact, despite their name, radiators transfer most of their heat to a room by convection.

9 Thermodynamic Basics

9.1 INTRODUCTION

9.1.1 What Is Thermodynamics?

Thermodynamics is the branch of physics that studies the effects of temperature and heat on physical systems at the macroscopic scale. In addition, it also studies the relationship that exists between heat, work and energy.

Thermal energy is found in many forms in today's society including power generation of electricity using gas, coal or nuclear, heating water by gas or electric, rocket propulsion, astronomy and cosmology, amongst many others.

9.1.2 Brief History

Engineering thermodynamics has its beginnings in the 1800s at the start of the industrial revolution in the UK when Thomas Newcomen (1712) designed and manufactured the first atmospheric steam engine, later improved by James Watt around 1764; this work laid the foundations for the development of the steam engine.

Although Thomas Savery (1650–1715) is acknowledged as the true inventor of the steam engine in 1698, he built the world's first commercially useful steam engine. It was originally intended for pumping water out of deep mines. The machine did not have a piston and relied on the opening and closing of various valves to direct the steam into the standing water, forcing it out in the exhaust.

In 1824, Sadi Carnot (1796–1832) published a paper entitled 'Reflections on the Motive Power of Fire' and this was cited to be the starting point for thermodynamics as a modern science. James Joule (1818–1889) together with Herman Helmholtz (1821–1894) independently formulated the relationship between heat and mechanical work. This discovery led to the theory of conservation of energy resulting in the first law of thermodynamics.

Josiah Willard Gibbs (1839–1903) is considered to be one of the founders of modern thermodynamics and a pioneer in graphical methods for thermal sciences and gives his name to the 'Gibbs Phase Rule'.

In 1862, James Clerk Maxwell put forward the philosophical argument that both light and radiant heat were forms of electromagnetic waves and this then led to the start of the quantitative analysis of thermal radiation. Jožef Stefan further observed that the total radiant flux from a black body is proportional to the fourth power of its temperature. This work was derived theoretically in 1884 by Ludwig Boltzmann and is known as the Stefan–Boltzmann law.

In later years, many workers have developed the theory of thermodynamics in a much wider field covering other sciences including astronomy and medical research.

9.2 BASIC THERMODYNAMICS

9.2.1 Basic Concepts

Properties are characteristics of a system which include mass, volume, energy, pressure and temperature. Thermodynamics also considers other quantities that are not physical properties, such as

TABLE 9.1
Symbols Used in Thermodynamics and Heat Transfer

Quantity	Units	Derived Units	S.I. Symbol
Absolute temperature	K		T
Adiabatic index			γ
Area	m^2		A
Celsius temperature	°C		θ
Characteristic gas constant	N m/kg K	J/kg K	R
Density	kg/m^3		ρ
Dynamic viscosity	$N\ s/m^2$	Pa s	η or μ
Energy	N m/kg K	Joule	
Enthalpy	N m	Joule	H
Entropy	J/K		S
Force	$kg\ m/s^2$	N	F
Heat transfer	N m	Joule	Q
Heat transfer rate	N m/s	Watt	Φ
Internal energy	N m	Joule	U
Kinematic viscosity	m^2/s		v
Length	m		Various
Mass	kg		m
Mass flow rate	kg/s		
Polytropic index			n
Pressure	N/m^2	Pascal	p
Pressure head	m		h
Specific enthalpy	N m/kg	J/kg	h
Specific entropy	J/kg K		s
Specific heat capacity	m/s	Joule/kg K	c
Specific internal energy	N m/kg	J/kg	u
Specific volume	m^3/kg		ν
Time	s		t
Universal gas constant	J/kmol K		R_o
Velocity	m/s^2		v or u
Volume	m^3		V or Q
Volumetric flow rate	m^3/s		
Weight	$kg\ m/s^2$	N	W
Work	N m	Joule	W
Work rate (power)	N/m/s	Watt	P

mass flow rates and energy transfers by work and heat. Properties are also considered to be either intensive or extensive.

Table 9.1 lists various quantities used in thermodynamics together with their units.

9.2.2 Extensive

An extensive property is one that is divisible in size; for example, volume when divided will become smaller. Mass and energy are other examples.

9.2.3 Intensive

Intensive properties are those that are independent of the size of the system; for example, temperature, pressure and density of a substance will remain unchanged when divided into smaller masses.

9.2.4 Specific and Total Quantities

These relate only to extensive properties.

A *specific quantity* represents in the case of, say, mass, the quantity per kg and is obtained by dividing the property by the mass. These properties are designated by lower case letters, such as 'v' for the specific volume (m^3/kg) and 'h' for specific enthalpy (J/kg).

Total quantity is always designated by higher case letters, say 'V' for volume (m^3) and 'H' for enthalpy (J).

Specific volume is principally used for gases and vapours. The inverse of a specific volume is density (ρ) (kg/m^3) and this property is chiefly used for solids and liquids but can also be used for gases ($\rho = 1/v$).

As the same letter is often used to designate more than one property, quite often alternative letters are used. As an example, 'v' for specific volume may occur in the same work as velocity, and 'u' and 'c' are sometimes used for velocity to avoid confusion in the case of initial velocity and final velocity.

9.2.5 Energy Forms

Fluids and solids can possess several forms of energy. All fluids possess energy due to their temperature and this is referred to as 'internal energy'. They will also possess 'gravitational or potential energy' (PE) due to distance (z) above a datum level and if the fluid is moving at a velocity (v), it will also possess 'kinetic energy'. If the fluid is pressurised, it will possess 'flow energy' (FE).

Pressure and temperature are the two governing factors and internal energy can be added to FE to produce a single property called 'enthalpy'.

In the following paragraphs, each of these terms will be considered in more detail.

9.2.6 Internal Energy

The molecules of a fluid possess both kinetic energy (KE) and PE relative to an internal datum. Generally, this is regarded simply as the energy due to its temperature and the change in internal energy in a fluid that undergoes a temperature change is given by

$$\Delta U = mc\Delta T \tag{9.1}$$

The total internal energy is denoted by the symbol 'U', which has values of J, kJ or MJ; also the specific internal energy 'u' has the values of kJ/kg.

Note: The change in temperature is in either degrees Celsius or Kelvin. The law which states internal energy is a function of temperature only; this is known as '*Joule's Law*' and is independent of its volume and pressure.

9.2.7 Gravitational or Potential Energy

When a mass 'm' kg is raised to a height of 'z' metres above a datum level, a lifting force is required. This force obviously has to be greater than the mass being lifted.

The work done in raising the mass is force × distance moved, so

$$\text{Work} = mgz \quad \text{where g is the gravity constant} \tag{9.2}$$

As energy has been used to do this work and cannot be destroyed, it will follow that energy has been stored in the mass. This energy is called the 'gravitational energy' or 'potential energy' (PE).

$$PE = mgz \tag{9.3}$$

9.2.8 Kinetic Energy

If a mass 'm' kg is at rest and is then accelerated 'a' to a velocity 'v' m/s, a force 'F' is required to provide the acceleration. This force is given by Newton's second law of motion:

$$F = m \cdot a \tag{9.4}$$

After a time 't' seconds, the mass will have travelled a distance of 'x' metres and have reached a velocity of 'v' m/s. These two quantities are related by the laws

$$a = \frac{v}{t} \tag{9.5}$$

$$x = \frac{v \cdot t}{2} \tag{9.6}$$

The work done in accelerating the mass

$$P = \frac{F}{A} \tag{9.7}$$

Energy has been expended in doing this work and has therefore been stored in the mass and will be carried along with it. This energy is called 'kinetic energy'.

$$KE = \frac{m \cdot v^2}{2} \tag{9.8}$$

9.2.9 Flow Energy

When fluid is pumped under pressure along a pipe or conduit, energy has been used to do the pumping. This energy is contained within the fluid and may be recovered as an example using a hydraulic cylinder.

Consider a piston pushing fluid within a cylinder. Fluid pressure is 'P' N/m^2 and the force required by the piston is

$$F = P \cdot A \tag{9.9}$$

where

P = fluid pressure
A = internal area of the pipe or conduit

When the piston has moved a distance of 'x' metres, the work done is

$$W = F \cdot x = P \cdot A \cdot x \tag{9.10}$$

Since $A \cdot x = V_{ol}$ and is the volume pumped in the cylinder, the work done is

$$W = PV_{ol} \tag{9.11}$$

Energy has been used in doing this work and it is stored in the fluid as 'flow energy':

$$FE = P \cdot V_{ol} \tag{9.12}$$

9.2.10 Enthalpy

As stated in Section 9.2.5, *Enthalpy* (H) requires both pressure and temperature; it therefore must possess both flow (FE) and internal energy (U). These two energies are added together

$$H = FE + U \tag{9.13}$$

The units are J for the total enthalpy or kJ/kg for the specific enthalpy.

9.2.11 Gas Laws

In this section, a '*perfect gas*' will be discussed and the following will be considered:

- Derive the basic gas laws.
- Derive the characteristic gas law.
- Examine the universal gas law.
- Define the 'mol'.

9.2.12 Theory

A gas is made up of molecules which are consistently moving around in a random manner. In a perfect gas, these molecules may collide but will have no tendency to join or repel each other. In reality, there is a small force of attraction between the gas molecules but as this force is so small, gas laws formulated for a perfect gas work quite well for a practical gas.

Each molecule possesses an instantaneous velocity and hence has KE. The sum of this energy is the internal energy 'U'. The velocity of the molecule will depend upon the temperature. When there is a change in the temperature, there will be a corresponding change in the internal energy. The internal energy is for all intents and purposes zero at –273°C. This temperature is known as absolute zero or 0 to K.

(Remember when converting from °Celsius to Kelvin to add 273 to the Celsius temperature.)

9.2.13 Pressure

When a fluid or a gas is compressed, it acquires pressure ('P'). Consider a gas within a closed container or vessel. The molecules bombard the inside of the container and each collision produces a momentum force. The force per unit area is the pressure of the gas:

$$P = \frac{F}{A} \tag{9.14}$$

where

F = force

A = area

The unit of pressure in SI units is N/m^2 or Pascals.

9.2.14 A Perfect Gas

A perfect gas is one in which its working temperature is well above its critical temperature and obeys the following:

- Boyle's law.
- Charles's law.
- Joule's law of internal energy.
- Dalton's law of partial pressures.
- Its specific heat is constant.

To obey these laws, the gas would

- Not change its state even at absolute zero.
- The distance between its molecules needs to be so far apart as not to have any intermolecular forces or collisions.
- At normal pressures and temperatures, the permanent gases such as oxygen, nitrogen, helium and so on closely obey these laws and are known as 'semi-perfect' gases.

9.2.15 Boyle's Law

Boyle's law states that provided the temperature 'T' of a perfect gas remains constant, the volume 'V' of a given mass of gas is inversely proportional to its pressure 'P' of the gas, that is, $P \propto 1/V$ (as shown in Figure 9.1) or $P \times V$ = constant if the temperature remains constant.

If the gas experiences a change in state during an isothermal process, then

$$P_1V_1 = P_2V_2 = \text{constant} \tag{9.15}$$

Representing the process on a graph having axes of pressure 'P' and volume 'V', the result will be as shown in Figure 9.2. The curve is known as a rectangular hyperbola and has the mathematical equation $x \cdot y$ = constant.

EXAMPLE 9.1

A certain perfect gas is heated at a constant temperature from an initial state of 0.22 m^3 and 325 kN/m^2 to a final state of 170 kN/m^2. Determine the final pressure of the gas.

Solution

State 1: $P_1 = 325\ kN/m^2$ and $V_1 = 0.22\ m^3$.
State 2: $P_2 = 170\ kN/m^2$ and $V_2 = ?$
From the equation, $P_1 \cdot V_1 = P_2 \cdot V_2$

$$V_2 = V_1 \times \frac{P_1}{P_2}$$

$$\frac{V_1}{T_1} = \frac{V_2}{T_2} = \text{constant}$$

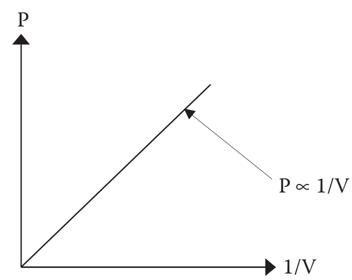


FIGURE 9.1 Graph of $P \propto 1/V$.

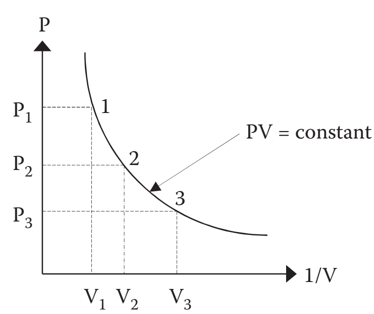


FIGURE 9.2 Graph of PV = constant.

$$V_2 = 0.22\,m^3 \times \frac{325\,kN/m^2}{170\,kN/m^2}$$

$$V_2 = 0.4206\ m^3$$

9.2.16 Charles's Law

Charles's law states that provided the pressure 'P' of a given mass of gas remains constant, the volume 'V' of the gas will be directly proportional to the absolute temperature 'T' of the gas, that is, $V \propto T$, or V = constant × 'T'. Therefore, V/T = constant for a constant pressure 'P'.

If the gas experiences a change in state during a constant pressure process, then

$$\frac{V_1}{T_1} = \frac{V_2}{T_2} = \text{constant} \tag{9.16}$$

This process is represented on a graph as shown in Figure 9.3.

EXAMPLE 9.2

A quantity of gas is subjected to a constant pressure process causing the volume of gas to reduce from 0.54 m³ at a temperature of 345°C to 0.32 m³. Calculate the final temperature of the gas at the end of this process.

Solution

From the question

$$V_1 = 0.54\ m^3$$

$$T_1 = 345 + 273\ K$$

$$V_2 = 0.32\ m^3$$

Now

$$\frac{V_1}{T_1} = \frac{V_2}{T_2} = \text{constant}$$

$$T_2 = T_1 \times \frac{V_2}{V_1}$$

$$= 618\,K\left(\frac{0.32\,m^3}{0.54\,m^3}\right)$$

$$T_2 = 366.22\,K$$

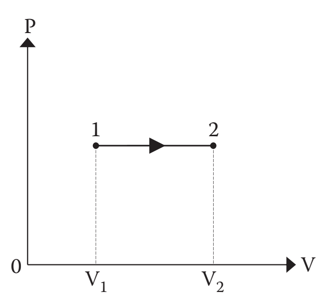


FIGURE 9.3 P–V graph of constant pressure.

9.2.17 Universal Gas Law

The universal gas equation combines pressure, volume and temperature and the relation between Boyle's and Charles's laws is expressed in Equation 9.17

$$\frac{P \cdot V}{T} = \text{constant} = R \tag{9.17}$$

where R is known as the universal gas constant. That is

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} \tag{9.18}$$

EXAMPLE 9.3

A volume of gas at a pressure of 325 kN/m^2 and temperature of 618 K is compressed to a volume 0.16 m^3 and a pressure of 380 kM/m^3. Determine the final temperature of the gas.

Solution

State 1: $P_1 = 325$ kN/m^3, $V_1 = 0.22$ m^3 and $T_1 = 618$ K.
State 2: $P_2 = 380$ kN/m^3, $V_2 = 0.16$ m^3 and $T_2 = ?$

From Equation 9.18:

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

$$T_2 = \frac{618\,\text{K} \times 380\,\text{kN/m}^2 \times 0.16\,\text{m}^3}{325\,\text{kN/m}^2 \times 0.22\,\text{m}^3}$$

$$T_2 = 525.52\ \text{K}$$

Gases, in practice, do not obey this law rigidly but many do tend towards it. This law is also referred to as the *characteristic equation of state of a perfect gas*.

The constant 'R' is called the gas constant and its units are Nm/kg K or J/kg K. Each perfect gas will have a different gas constant.

The characteristic gas equation is usually written as

$$PV = RT \tag{9.19}$$

or for 'm' kg, occupying V m^3

$$PV = mRT \tag{9.20}$$

The characteristic gas equation can be expressed in a different form which is derived using the kilogramme-mole as a unit. The kilogramme-mole is defined as a quantity of gas equivalent to m kg of the gas, where M is the molecular weight of the gas. Oxygen has a molecular weight of 32; hence, 1 kilogramme-mole of oxygen is equivalent to 32 kg of oxygen.

From the definition of the kilogramme-mole, for 'm' kg of the gas

$$m = nM \tag{9.21}$$

where n is the number of moles.

Note well: As the standard of mass in SI units is the kilogramme (kg), the kilogramme-mole will be written as 'mole'.

Substituting for m from Equation 9.21 in Equation 9.20

$$P \cdot V = nMRT \quad \text{or} \quad MR = \frac{P \cdot V}{nT} \tag{9.22}$$

Avogadro's hypothesis states that the volume of 1 mole of any gas will be the same as the volume of 1 mole of any other gas when the gases are at the same temperature and pressure. Therefore, V/n is the same for all gases at the same values of P and T.

The quantity PV/nT is a constant for all gases. This constant is called the 'universal gas constant' and has the symbol 'R_o'.

$$MR = R_o = \frac{PV}{nT} \tag{9.23}$$

or

$$PV = nR_oT \tag{9.24}$$

As $MR = R_o$, then

$$R = \frac{R_o}{M} \tag{9.25}$$

It has been shown by experiment that the volume of 1 mole of any perfect gas at a pressure of one bar and at a temperature of one degree C is approximately 22.71 m^3.

From Equation 9.24:

$$R_o = \frac{PV}{nT}$$

$$= \frac{1 \times 10^5 \times 22.71}{1 \times 273.15}$$

$$= 8314.11 \text{ J/mole K}$$

The gas constant can be found for any gas when the molecular weight is known using the Equation 9.25. The molecular weight for oxygen is 32; hence, the gas constant is

$$R = \frac{R_o}{M}$$

$$= \frac{8314.11}{32}$$

$$= 259.81 \text{ J/kg K}$$

EXAMPLE 9.4

The volume of a gas at 0.046 m^3 is contained in a sealed cylinder under a pressure of 300 kN/m^2 at a temperature of 45°C. The gas is then compressed until a pressure of 1.27 MN/m^2 at a temperature of 83°C is reached. Assuming the gas is a perfect gas, given R = 0.29 kJ/kg K, calculate

1. The mass of the gas (kg)
2. The final volume of the gas (m^3)

Solution

State 1: $V_1 = 0.046\ m^3$, $P_1 = 300\ kN/m^2$ and $T_1 = 273 + 45.0\ K = 318.0\ K$.
State 2: $P_2 = 1.27\ MN/m^2$, $T_2 = 273 + 83.0\ K = 356.0\ K$ and $R = 0.29\ kJ/kg\ K$.

1. From Equation 9.20:

$$PV = mRT$$

$$m = \frac{P_1V_1}{RT_1}$$

$$= \frac{300 \times 0.046}{0.29 \times 318}$$

$$= 0.1496\ kg$$

2. From Equation 9.16:

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$

$$V_2 = V_1\left(\frac{T_2}{T_1}\right)$$

$$V_2 = 0.046\ m^3\left(\frac{356\ K}{318\ K}\right)$$

$$V_2 = 0.051\ m^3$$

9.2.18 Specific Heat Capacity

The specific heat capacity of any substance is defined as the amount of energy required to raise a unit mass through one degree temperature rise. In thermodynamics, there are two specified conditions used:

1. Constant volume (C_v)
2. Constant pressure (C_p)

The two specific heat capacities do not have the same value, and it is very important to distinguish between them.

9.2.19 Specific Heat Capacity at Constant Volume (C_v)

Consider one kg of a gas supplied with an amount of heat energy sufficient to raise its temperature by 1 K while the volume of the gas constant, the amount of heat energy supplied, is known as the specific heat capacity at constant volume and is denoted by C_v. The basic unit of C_v is J/kg K.

For a reversible non-flow process at constant volume:

$$dQ = mC_v dT \tag{9.26}$$

For a perfect gas, the value of C_v will be constant for any one gas at all pressures and temperatures. Equation 9.26 can be expanded as follows.

Heat flow in a constant volume process between two states:

$$Q_{12} = mC_v(T_2 - T_1) \tag{9.27}$$

From the non-flow energy equation:

$$Q - W = (U_2 - U_1)$$

$$mC_v(T_2 - T_1) - 0 = (U_2 - U_1)$$

$$(U_2 - U_1) = mC_v(T_2 - T_1) \tag{9.28}$$

that is

$$dU = Q \tag{9.29}$$

Note: In a reversible constant volume process, there will be no work energy transfer as the piston will be unable to move; therefore, W = 0.

Figure 9.4 shows the P–V diagram for the reversible constant volume process.

EXAMPLE 9.5

A quantity of 4.5 kg of gas is heated at a constant volume of 1.5 m^3 and temperature 20°C until the temperature rose to 150°C. If the gas is assumed to be perfect, determine

1. The heat flow during the process
2. The pressure at the beginning of the cycle
3. The final pressure

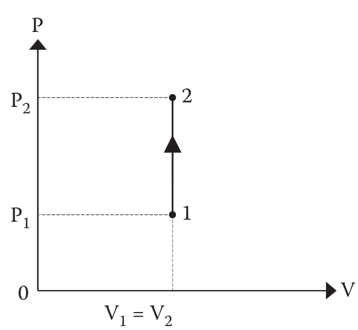


FIGURE 9.4 P–V graph for the reversible constant volume process.

Given:

$C_v = 0.72$ kJ/kg K and $R = 0.287$ kJ/kg K.

Solution

m =	4.5	kg
V_1 =	1.5	m^3
V_2 =	1.5	m^3
T_1 =	20 + 273 = 293	K
T_2 =	150 + 273 = 423	K
C_v =	0.72	kJ/kg K
R =	0.287	kJ/kg K

1. From Equation 9.27:

$$Q_{12} = mC_v(T_2 - T_1)$$

$$= 4.5 \text{ kg} \times 0.72 \text{ kJ/kg K} \times (423 - 293) \text{ K}$$

$$= 421.2 \text{ kJ}$$

2. From Equation 9.20, PV = mRT

   For state 1:

$$P_1V_1 = mRT$$

$$P_1 = \frac{mRT_1}{V_1}$$

$$P_1 = \frac{4.5 \text{ kg} \times 0.287 \text{ kJ/kg K} \times 293 \text{ K}}{1.5 \text{ m}^3}$$

$$P_1 = 252.27 \text{ kN/m}^2$$

3. For state 2:

$$P_2V_2 = mRT$$

$$P_2 = \frac{mRT}{V_2}$$

$$P_2 = \frac{4.5 \text{ kg} \times 0.287 \text{ kJ/kg K} \times 423 \text{ K}}{1.5 \text{ m}^3}$$

$$P_2 = 364.20 \text{ kN/m}^2$$

9.2.20 Specific Heat Capacity at Constant Pressure (C_p)

When 1 kg of a gas is supplied with an amount of heat energy sufficient to raise the temperature by 1 K while the pressure of the gas remains constant, the amount of heat energy that is supplied is known as the specific heat capacity at constant pressure and is denoted by C_p. The unit of C_p is J/kg K.

For a reversible non-flow process at constant pressure:

$$dQ = mC_p dT \tag{9.30}$$

For a perfect gas, the value of C_p is constant for any one gas at all pressures and temperatures. Equation 9.20 can be expanded as follows:

In a reversible constant pressure process, the heat flow

$$Q = mC_p(T_2 - T_1) \tag{9.31}$$

9.2.21 Relationship between the Specific Heats

Consider a perfect gas being heated at constant pressure from T_1 to T_2. Referring to the non-flow equation, $Q = U_2 - U_1 + W$ and the equation for a perfect gas, $U_2 - U_1 = mC_v(T_2 - T_1)$, combining will give

$$Q = mC_v(T_2 - T_1) + W$$

In a constant pressure process, the work done by the fluid is given by

$$W = P \cdot \Delta V$$

that is

$$W = P(V_2 - V_1) \tag{9.32}$$

Using the equation $PV = mRT$:

$$W = mR(T_2 - T_1) \tag{9.33}$$

Substituting:

$$Q = mC_v(T_2 - T_1) + mR(T_2 - T_1)$$

$$= m(C_v + R)(T_2 - T_1) \tag{9.34}$$

Equating (9.31) and (9.34) for the heat flow ‘Q’:

$$mC_p(T_2 - T_1) = m(C_v + R)(T_2 - T_1)$$

Therefore,

$$C_p = C_v + R \tag{9.35}$$

This equation may also be written as

$$R = C_p - C_v \tag{9.36}$$

9.2.22 Specific Heat Ratio 'γ'

The ratio of specific heat at constant pressure to the specific heat at constant volume is given by the symbol 'γ' (gamma).

$$\gamma = \frac{C_p}{C_v} \tag{9.37}$$

From Equation 9.25, it is clear that C_p has to be greater than C_v for a perfect gas. It follows, therefore, that the ratio $C_p/C_v = \gamma$ is always greater than unity. In general, 'γ' is approximately 1.4 for diatomic gases such as carbon monoxide (CO), hydrogen (H_2), nitrogen (N_2) and oxygen (O_2). For monatomic gases such as argon (A) and helium (He), γ is approximately 1.6 and for triatomic gases including carbon dioxide (CO_2) and sulphur dioxide (SO_2), γ is about 1.3.

Some useful relationships between C_p, C_v R and γ can be derived from Equations 9.36 and 9.37.

$$C_p - C_v = R$$

Dividing through by 'C_v':

$$\frac{C_p}{C_v} - 1 = \frac{R}{C_v}$$

Now from Equation 9.57:

$$\gamma = \frac{C_p}{C_v}$$

$$\gamma - 1 = \frac{R}{C_v}$$

$$C_v = \frac{R}{(\gamma - 1)} \tag{9.38}$$

Also from Equation 9.37, $C_p = \gamma C_v$ and substituting in Equation 9.38:

$$C_p = \gamma C_v = \frac{\gamma R}{(\gamma - 1)}$$

$$C_p = \frac{\gamma R}{(\gamma - 1)} \tag{9.39}$$

EXAMPLE 9.6

A particular perfect gas has a specific heat as follows:

$$C_p = 0.846 \text{ kJ/kg K} \quad \text{and} \quad C_v = 0.657 \text{ kJ/kg K}$$

Determine the gas constant and the molecular weight of the gas.

Solution

From Equation 9.36

$$R = C_p - C_v$$

that is

$$R = 0.846 - 0.657$$
$$= 0.189 \text{ kJ/kg K}$$

or R = 189 Nm/kg K.

From Equation 9.25

$$M = \frac{R_o}{R}$$

that is

$$M = \frac{8314.4}{189}$$

$$= 44.0 \text{ kg/kmol}$$

9.3 LAWS OF THERMODYNAMICS

9.3.1 Conservation of Energy

The law of conservation of energy was first formulated in the nineteenth century.

This law states that the total energy of an isolated system remains constant regardless of changes within the system. An alternative way of stating this is that energy is a quantity that can be converted from one form to another, but cannot be created nor destroyed.

It was Gottfried Wilhelm Leibniz during the period from 1676 to 1689 who first attempted a mathematical formulation of the kind of energy which is connected with motion (KE). Leibniz noticed that in many mechanical systems (of several masses, m_i each with velocity v_i), $\sum_i m_i v_i^2$ was conserved so long as the masses did not interact. He called this quantity the vis viva or living force of the system. The principle represents an accurate statement of the approximate conservation of KE in situations where there is no friction.

9.3.2 First Law of Thermodynamics

From the Equation 9.40, it is seen that in terms of heat flow and work output into and out of the engine, the system is in a state of balance. This state is known as the 'steady state' and can be explained by the '*First Law of Thermodynamics*' which states that the first law is a consequence of energy and requires that a system may exchange energy with its surroundings strictly by heat flow or work.

Thus,

$$\Delta E = \Delta Q - \Delta W \tag{9.40}$$

where

ΔE is the change in energy
ΔQ is the change in heat
ΔW is the change in work

9.3.3 Steady Flow Process

The process which changes the state of a thermodynamic system can be divided into two main types depending upon whether the system is an open or closed system.

9.3.4 Flow Process

A flow process is known as an open system where work energy, heat energy and the working fluid may be transferred across the boundary (Figure 9.5).

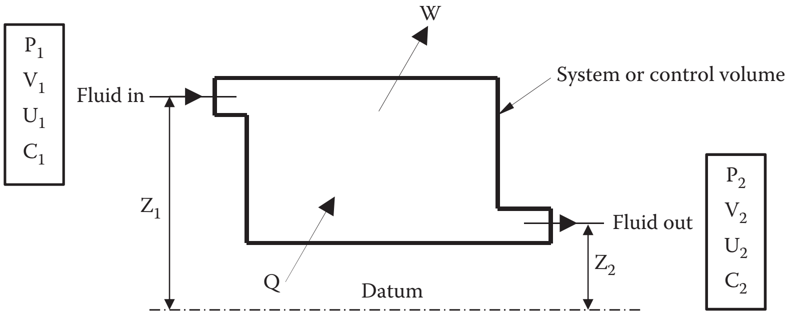


FIGURE 9.5 Thermodynamic process.

General energy equation

$$gZ_1 + U_1 + p_1V_1 + \frac{C_1^2}{2} + Q = gZ^2 + U^2 + p_2V_2 + \frac{C_2^2}{2} + W \qquad (9.41)$$

where
gZ_1 = potential energy
U = internal energy

$$\frac{C^2}{2} = \text{kinetic energy}$$

Equation 9.40 may be written as

$$gZ_1 + h_1 + \frac{C_1^2}{2} + Q = gZ_2 + h_2 + \frac{C_2^2}{2} + W \qquad (9.42)$$

where U + pV = h = *enthalpy*.

Enthalpy is a measure of the total energy of a thermodynamic system. It includes the internal energy, which is the energy required to create a system, and the amount of energy required to make room for it by displacing its environment and establishing its volume and pressure.

Enthalpy is a thermodynamic potential. It is a state function and an extensive quantity. The unit of measurement for enthalpy in the International System of Units (SI) is the joule.

9.3.5 Consider a Boiler at Constant Pressure

Figure 9.6 depicts a boiler supplying steam with water and heat being supplied.

Applying the general energy equation:

- PE is small when compared with the heat energy being supplied and therefore can be neglected.
- KE is also very small and is neglected.
- As there are no working parts, 'W' = 0.

The equation can now be written in the following manner

$$h_1 + Q = h_2 \qquad (9.43)$$

$$Q = h_2 - h_1 \quad \text{(for Mkg flow of fluid)}$$

$$\therefore \quad Q = M(h_2 - h_1) \qquad (9.44)$$

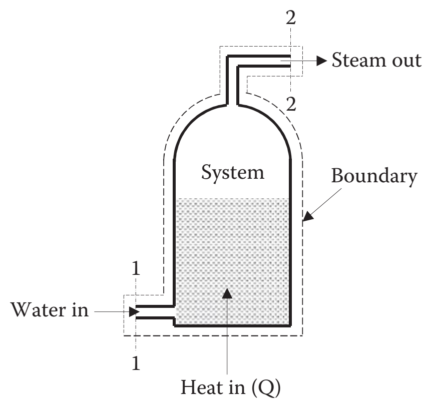


FIGURE 9.6 Boiler supplying steam.

EXAMPLE 9.7

A boiler operating at a constant pressure of 1.5 MPa evaporates fluid at the rate of 1000 kg/h. At the boiler inlet, the entering fluid has an enthalpy of 165 kJ/kg. At the exit to the boiler, the enthalpy of the exiting steam is 2200 kJ/kg. Calculate the heat energy required by the boiler.

Solution

From the general energy equation

$$Q - W = M\left[(h_2 - h_1) + \left(\frac{C_2^2 - C_1^2}{2}\right)\right]$$

1. For the boiler, $(C_2^2 - C_1^2)/2$ is assumed to be zero.
2. No working or moving parts; therefore, $W = 0$.

Where

Q = heat energy per hour required by the boiler
W = work energy leaving the system = 0
m = fluid flow rate = 1000 kg/h
h_1 = enthalpy at inlet = 165 kJ/kg
h_2 = enthalpy at outlet = 2200 kJ/kg

Hence,

$$Q = 1000(2200 - 165)\ \text{kJ/h}$$
$$Q = 2.035 \times 10^6\ \text{kJ/h}$$

If 60% of the heat energy supplied to the boiler is used in evaporating the fluid, calculate the rate of fuel consumption required to maintain the rate of evaporation when 1 kg of fuel produces 30,000 kJ of heat energy.

$$\text{Heat energy required per hour} = \frac{2.035 \times 10^6}{0.60}$$
$$= 3.392 \times 10^6\ \text{kJ/h}$$

The heat energy available from the fuel

$$= 30,000\ \text{kJ/h}$$

Hence

$$\text{Fuel required} = \frac{3.392 \times 10^6}{30{,}000}\frac{\text{kJ}}{\text{h}} \times \frac{\text{kg}}{\text{kJ}}$$
$$= 113.1\,\text{kg/h}$$

EXAMPLE 9.8

A condenser has fluid flowing through it at the rate of 35 kg/min. The fluid enters with a specific enthalpy of 2500 kJ/kg and leaves with a specific enthalpy of 250 kJ/kg. Calculate the rate of heat energy loss from the system.

Solution

The steady FE equation gives

$$Q - W = \dot{m}\left[(h_2 - h_1) + \left(\frac{C_2^2 - C_1^2}{2}\right)\right]$$

For a condenser, W = 0 and the terms representing the change in KE and PE may be neglected. The equation will reduce to

$$Q = \dot{m}(h_2 - h_1)$$

Hence,

$$Q = 35\frac{\text{kg}}{\text{min}}(250 - 2500)\frac{\text{kJ}}{\text{kg}}$$

$$Q = -78{,}750\ \text{kJ/min}$$

9.3.6 Nozzle

A nozzle is an efficient way to convert thermal energy into KE.

Figure 9.7 depicts a convergent nozzle with a mass flow of fluid passing through it. The steady FE equation gives

$$Q - W = \dot{m}\left[(h_2 - h_1) + \left(\frac{C_2^2 - C_1^2}{2}\right) + (gZ_2 - gZ_1)\right]$$

- The average velocity of the fluid through the nozzle will be high and only spends a short duration within the nozzle. As a result, it is considered that there is not sufficient time for any heat energy to flow either into or out of the fluid; as a result, it is considered that 'Q' = 0.

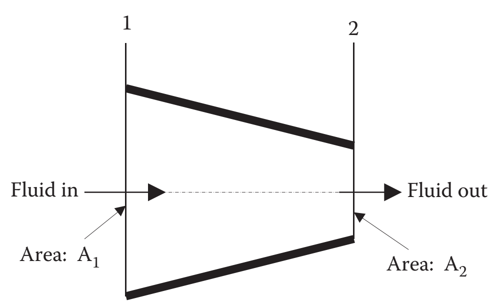


FIGURE 9.7 Mass flow through a convergent nozzle.

- As a nozzle has no moving parts, hence no work energy will be transferred to or from the fluid as it passes through the nozzle; therefore, 'W' = 0.
- PE is generally small and can be neglected.

The equation becomes

$$0 = \dot{m}\left[(h_2 - h_1) + \left(\frac{C_2^2 - C_1^2}{2}\right)\right] \quad (9.45)$$

Often C_1 is negligible compared with C_2, and in this case the equation will reduce to

$$0 = \dot{m}\left[(h_2 - h_1) + \left(\frac{C_2^2}{2}\right)\right] \quad (9.46)$$

or

$$\frac{C_2^2}{2} = (h_1 - h_2) \quad (9.47)$$

Equation 9.47 can also be rewritten as

$$C_2 = \sqrt{2(h_1 - h_2)} \quad (9.48)$$

EXAMPLE 9.9

Fluid flowing through a horizontal nozzle has the following properties:

1. At the inlet to the nozzle, the fluid has a specific enthalpy of 2800 kJ/kg with negligible velocity.
2. At its outlet, the fluid has a specific enthalpy of 2250 kJ/kg and its specific volume is 1.25 m^3/kg.

Calculate the required area of the nozzle assuming the flow is adiabatic.

SOLUTION

From the steady FE equation:

$$Q - W = \dot{m}\left[(h_2 - h_1) + \left(\frac{C_2^2 - C_1^2}{2}\right) + (gZ_2 - gZ_1)\right]$$

When this equation is applied to the nozzle, it becomes

$$0 = \dot{m}\left[(h_2 - h_1) + \left(\frac{C_2^2 - C_1^2}{2}\right)\right]$$

As the inlet C_1 is negligible, the equation can be rewritten as

$$\begin{aligned} C_2 &= \sqrt{2(h_1 - h_2)} \\ &= \sqrt{[2(2800 - 2250)]} \\ &= 33.166 \text{ m/s} \end{aligned}$$

Applying the equation of continuity at the outlet to the nozzle gives

$$\dot{m} = \frac{A_2 C_2}{v_2}$$

Rearranging to solve for 'A_2':

$$A_2 = \frac{\dot{m} \cdot v_2}{C_2}$$

$$A_2 = \frac{14 \text{ kg/s} \times 1.25 \text{ m}^3/\text{kg}}{33.166 \text{ m/s}}$$

$$A_2 = 0.5276 \text{ m}^2 \quad \text{(Area of exit of convergent nozzle)}$$

The respective diameter will be

$$d_2 = \sqrt{\frac{4 \cdot A_2}{\pi}}$$

$$d_2 = 0.8196 \text{ m}$$

9.3.7 Pump

A pump is a device that uses external work energy to produce a pressure rise in a flowing fluid.

Figure 9.8 portrays a pump defining its boundary with its input and output together with its losses. The steady FE equation gives

$$Q - W = \dot{m}\left[(h_2 - h_1) + \left(\frac{C_2^2 - C_1^2}{2}\right) + (gZ_2 - gZ_1)\right]$$

- Although the velocities of the fluid will be high, the differences between the input and output velocities through the pump will not be large. The term representing the change in KE can be disregarded.
- PE will be generally small and can also be neglected.
- 'W' is the amount of work energy/second required to drive the pump.

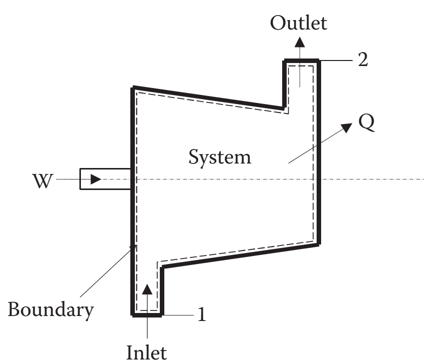


FIGURE 9.8 Pump.

The steady state FE equation becomes

$$-W = \dot{m}(h_2 - h_1) \tag{9.50}$$

as $h_2 > h_1$, 'W' will be found to be negative.

EXAMPLE 9.10

Fluid is being pumped at the rate of 45 kg/min. The specific enthalpy of the fluid at the inlet to the pump is 46 kJ/kg and at the outlet of the pump it is 175 kJ/kg. Assuming 105 kJ/kg of heat energy is lost to its surroundings by the pump, calculate the power required to drive the pump assuming the pump efficiency is 85%.

Solution

The fluid flow rate, '$\dot{m}$' = 45 kg/min

= 0.75 kg/s

Heat loss 'Q' = –105 kJ/min

= –1.75 kJ/s

Specific enthalpy (h_1) at inlet = 46 kJ/kg and specific enthalpy (h_2) at exhaust = 1.27 kJ/kg.

The kinetic and potential energies are neglected. Substituting the above data into the steady FE equation

$$-1.75 - W = 0.75 \times (175 - 46) \text{ kJ/s}$$

Hence,

$$W = -1.75 - (0.75 \times 129)$$

$= -98.5$ kJ/s

$= -98.5$ kW (the negative sign indicating this is work energy required to drive the pump)

As the pump efficiency is 85%, power required to drive the pump

= 98.5/0.85

= 115.88 kW

9.3.8 Turbine

A turbine is opposite to that of a pump in that a pressure drop across the turbine results in work energy being produced (see Figure 9.9).

The steady FE equation gives

$$Q - W = \dot{m}\left[(h_2 - h_1) + \left(\frac{C_2^2 - C_1^2}{2}\right) + (gZ_2 - gZ_1)\right]$$

- The average velocity flow of fluid through a turbine is normally high and passes quickly through the turbine. It can be assumed that heat energy does not have time to flow either into or out of the fluid, and hence Q = 0.
- The velocities through the turbine are high and the differences between the inlet velocity and outlet velocity are not large; therefore, the term representing the change in KE can be neglected.

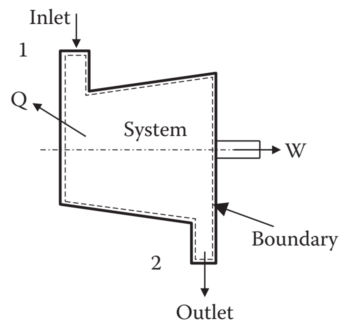


FIGURE 9.9 Turbine.

- PE is generally small and can be neglected.
- 'W' is the amount of external work energy produced per second.

The steady FE equation can therefore be reduced to

$$-W = \dot{m}(h_2 - h_1) \tag{9.50}$$

or

$$W = \dot{m}(h_1 - h_2) \tag{9.51}$$

EXAMPLE 9.11

In a turbine, fluid flows through it at the rate of 45 kg/min. The specific enthalpy drop of the fluid is 580 kJ/kg. The turbine loses 2100 kJ/min in the form of heat energy. Calculate the power produced by the turbine assuming the changes in potential and KE are small and can be neglected.

Solution

From the question, Q = heat energy flow into the system = –2100 kJ/min, W = work energy flow from the system kJ/min, m = fluid flow rate = 45 kg/min, $h_2 - h_1 = -580$ kJ/kg, C_1 and C_2 = neglected and Z_1 and Z_2 = neglected.

The steady FE equation becomes

$$\begin{aligned} -2100\text{ kJ/min} - W &= 45\text{ kg/min} \times -580\text{ kJ/kg} \\ W &= (26{,}100 - 2100)\text{ kJ/min} \\ &= 24{,}000\text{ kJ/min} \\ &= 400\text{ kJ/s} \\ &= 400\text{ kW} \end{aligned}$$

9.3.9 Throttling

A throttling process is where the fluid is made to flow through a restriction; for example, an orifice or a partially open throttling valve. This will cause a substantial pressure drop in the fluid within the vicinity of the restriction. Throttling is a non-reversible flow process.

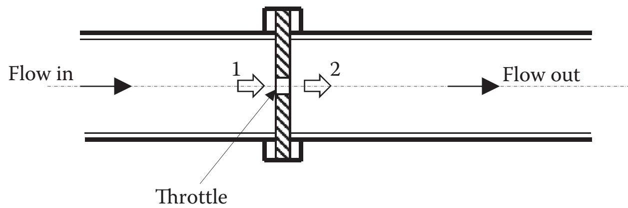


FIGURE 9.10 Throttling process.

Figure 9.10 shows a pipe fitted with an orifice plate. One reason for fitting an orifice plate in a pipeline is to measure the pressure within a fluid passing through the pipe.

The general steady FE equation

$$Q - W = M\left[(h_2 - h_1) + \left(\frac{C_2^2 - C_1^2}{2}\right) + (gZ_2 - gZ_1)\right]$$

- PE can be considered to be small enough to be neglected.
- There are no working parts; no energy can be transferred in the form of work energy; therefore, $W = 0$.
- The throttling takes place over a very small distance and the available area through which heat energy can flow either into or from the fluid is very small, so it can be assumed that very little energy is lost by heat transfer; hence, $Q = 0$.
- Any difference that exists between C_1 and C_2 will be small, and as a result any change in the KE is normally not considered.

The steady FE equation becomes

$$0 = \dot{m}(h_2 - h_1) \tag{9.52}$$

that is, during a throttling process, the enthalpy will remain constant.

EXAMPLE 9.12

A fluid flowing through a pipeline undergoes a throttling process where the pressure is reduced from 10 bar to 1 bar when passing through an orifice plate. Prior to throttling, the specific volume of the fluid is 0.3 m^3/kg, and after throttling it is 1.8 m^3/kg. Calculate the change in specific internal energy during this throttling process.

Solution

For a throttling process, the steady state FE equation can be written as

$$0 = \dot{m}(h_2 - h_1) \quad \text{or} \quad h_2 - h_1$$

But $h_2 = u_2 + P_2v_2$ and $h_1 = u_1 + P_1v_1$.

The change in specific internal energy

$$\begin{aligned}
&= u_2 - u_1 \\
&= (h_2 - P_2v_2) - (h_1 - P_1v_1) \\
&= (h_2 - h_1) - (P_2v_2 - P_1v_1) \\
&= 0 - (1 \times 1.8 - 10 \times 0.3) \text{ bar m}^3\text{/kg} \\
&= 120 \times 10 \text{ Nm/kg} \\
&= 120 \text{ kJ/kg}
\end{aligned}$$

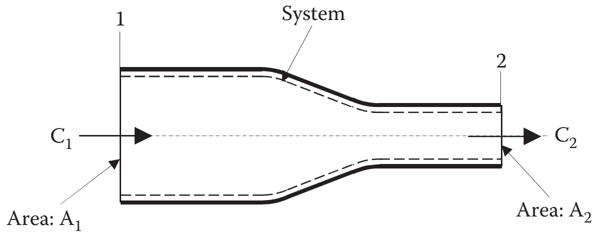


FIGURE 9.11 Mass flow through a system.

9.3.10 Equation of Continuity

Consider any section of the cross-sectional area 'A', where the fluid velocity is 'C'; then the rate of volume flow past the section is 'CA'. Also, since the mass flow is volume flow divided by the specific volume

$$\text{Mass flow rate, } \dot{m} = \frac{CA}{\nu}$$

where ν = specific volume at the particular section

With reference to Figure 9.11

$$\dot{m} = \frac{C_1 \cdot A_1}{\nu_1} = \frac{C_2 \cdot A_2}{\nu_2} \tag{9.53}$$

9.3.11 Non-Flow Processes

In the previous section, the discussion centred on a fluid flowing through a system. In this section, the discussion will be where the fluid undergoes a series of processes where it does not flow. An example of this is the cylinder of an internal combustion engine. Referring to Figure 9.12 in the suction stroke, the working fluid is drawn into the cylinder where it is temporarily sealed. Whilst the cylinder is sealed, the fluid is compressed by the rising piston. Thermal energy is then introduced which produces sufficient energy forcing the piston back down the cylinder, thereby creating external work. On completion of the working stroke, the exhaust valve is opened and the fluid is forced out of the cylinder back into its surroundings. The processes which are subject to action where the working fluid cannot cross the system boundary are called *non-flow processes*. This process occurred during the compression and working strokes in the previous example.

Referring to the general energy equation in Equation 9.41

$$gZ_1 + U_1 + P_1V_1 + \frac{C_1^2}{2} + Q = gZ^2 + U^2 + P_2V_2 + \frac{C_2^2}{2} + W$$

If the fluid undergoes a non-flow process from state (1) to state (2), the terms from the general energy equation for p_1V_1 and p_2V_2 will be zero as the fluid is already in the system and will still be in the system at the end of the process. For similar reasons, the changes in kinetic and potential energies of the fluid will also be zero. The general energy equation will reduce to

$$U_1 + Q = U_2 + W \tag{9.54}$$

or

$$U_2 - U_1 = Q - W \tag{9.55}$$

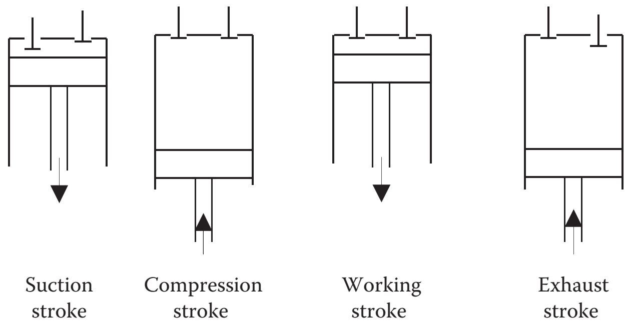


FIGURE 9.12 Internal combustion engine cycle.

that is, in a non-flow process, the change in internal energy of the fluid is equal to the net amount of heat energy supplied to the fluid minus the net amount of work energy flowing from the fluid.

This equation is known as the *non-flow energy equation* and in the following discussion it will be shown how this is applied to the various non-flow processes.

9.3.12 Constant Temperature (Isothermal) Process (pV – C)

In a process where the change in temperature is very small, that process can be approximated to an isothermal process. As an example, consider the slow expansion or compression of a working fluid in a cylinder that is perfectly cooled by water; this process may be analysed assuming the temperature remains constant.

Referring to Figure 9.13a and b which shows a constant temperature (isothermal) process, the general relationship of the properties between the initial and final states of a perfect gas will be as follows:

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2} \tag{9.56}$$

If the temperature remains constant throughout the process, that is, $T_1 = T_2$, then the above relationship becomes

$$p_1 V_1 = p_2 V_2 \tag{9.57}$$

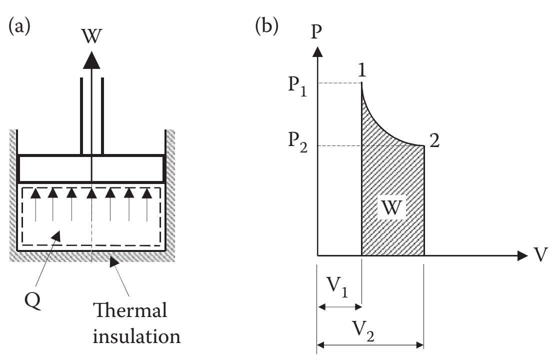


FIGURE 9.13 (a) and (b): Constant temperature (isothermal) process.

From this equation, it will be seen that an increase in the volume will result in a decrease in the pressure. In other words, in an isothermal process, the pressure is inversely proportional to its volume.

9.3.12.1 Work Transfer

Referring to the process represented on the p–V diagram in Figure 9.13, it is noted that the volume increases during the process, that is, the fluid is expanding and the expansion work is given by

$$W = \int_1^2 p\,dV$$

$$= \int_1^2 \frac{c}{V}\,dV \quad (pV = c\text{, a constant})$$

$$= c\int_1^2 \frac{dV}{V}$$

$$= p_1V_1 \ln\frac{V_2}{V_1}$$

$$= mRT_1 \ln\frac{V_2}{V_1} \quad (p_1V_1 = mRT)$$

$$= mRT_1 \ln\frac{p_1}{p_2} \quad \left(\text{since } \frac{V_2}{V_1} = \frac{p_1}{p_2}\right) \tag{9.58}$$

On the p–V diagram shown in Figure 9.13b, the shaded area under the curve represents the amount of work being transferred from the system.

As this is an expansion process (that is an increasing volume), the work is being done by the system, that is, the system produces work output and this is shown by the direction of the arrow representing 'W'.

9.3.12.2 Heat Transfer

Applying the heat balance to this case

$$U_1 + Q = U_2 + W \text{ (from Equation 9.54)}$$

For a perfect gas:

$$U_1 = mc_vT_1 \quad \text{and} \quad U_2 = mc_vT_2$$

As the temperature is constant:

$$U_1 = U_2$$

Substituting in the energy balance equation:

$$Q = W \tag{9.59}$$

Note that the heat flow is equivalent to the work done in an isothermal process *for a perfect gas only*. It should also be pointed out that on the p–V diagram when the process takes place from right to left, the work done by the fluid is negative. That is, work is being done on the fluid.

EXAMPLE 9.14

A process compresses 0.4 kg of oxygen isothermally from 1.01 bar to 5.5 bar at 22°C. Establish the work done during the compression and the heat transferred during the process. Assume that the oxygen is a perfect gas and the molecular weight of the oxygen is M = 32 kg/kmol.

Solution

From the question, m = 0.4 kg, p_1 = 1.01 bar, T_1 = 22°C, p_2 = 5.5 bar and M = 32 kg/kmol.

$$\text{From the equation } R = \frac{R_o}{M}$$

$$= \frac{8314}{32}$$

$$= 0.260 \text{ kJ/kg K}$$

For an isothermal process, work input

$$W = mRT_1 \ln\frac{p_2}{p_1} \quad \text{(from Equation 9.58)}$$

$$= 0.4 \times 0.260 \times (22 + 273) \ln\frac{5.5}{1.01}$$

$$= 52 \text{ kJ}$$

As this is an isothermal process, all the work input is rejected as heat; hence, Q = W = 52 kJ.

9.3.13 Adiabatic Process (Q = 0)

An adiabatic process is one in which there is no heat transferred to or from the fluid during the process. Such a system is thermally isolated and the process within such a system may be idealised as an adiabatic process. Such a process can be either reversible or irreversible. Only the reversible process will be considered here.

Steam engine cylinders and gas turbine casings are generally well insulated to minimise any heat loss from the system. The fluid expansion process in such cases can be assumed to be adiabatic.

Figure 9.14a and b depicts an adiabatic process (zero heat transfer).

For a perfect gas, the equation for an adiabatic process is

$$pV^{\gamma} = C \tag{9.60}$$

where γ = ratio of specific heat = C_p/C_v.

Equation 9.60 applies to states 1 and 2 as

$$p_1V_1^{\gamma} = p_2V_2^{\gamma} \tag{9.61}$$

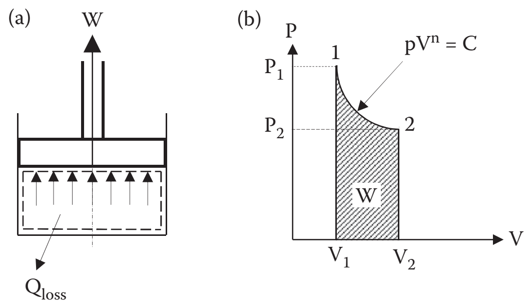


FIGURE 9.14 (a) and (b): Polytropic process.

Equation 9.61 can be rewritten as

$$\frac{p_2}{p_1} = \left[\frac{V_1}{V_2}\right]^{\gamma} \tag{9.61a}$$

By manipulating Equations 9.56 and 9.61, the following relationship can be determined:

$$\frac{T_2}{T_1} = \left[\frac{p_2}{p_1}\right]^{\frac{\gamma-1}{\gamma}} = \left[\frac{V_1}{V_2}\right]^{\gamma-1} \tag{9.62}$$

Examining Equations 9.61 and 9.62 for adiabatic processes on a perfect gas, the following conclusions can be made.

- Any increase in pressure will result in an increase in temperature.
- Any increase in volume will result in a decrease in pressure.
- Any increase in volume will result in a decrease in temperature.

9.3.13.1 Work Transfer

A reversible adiabatic process for a perfect gas is shown on the p–V diagram in Figure 9.14b and it is noted that the volume increases during the process. The work done is given by the shaded area and this area can be evaluated by integration—that is

$$W = \int_1^2 p\, dV \tag{9.63}$$

Since $pV^{\gamma} = c$, a constant, then

$$= \int_1^2 \frac{c}{V^{\gamma}} dV$$

$$= c\int_1^2 \frac{dV}{V^{\gamma}}$$

$$W = \frac{p_1V_1 - p_2V_2}{\gamma - 1} \tag{9.64}$$

Note that after expansion, p_2 is smaller than p_1.

As this is an expansion process, the work is done by the system, that is, the system produces work output and this is shown by the direction of the arrow representing ‘W’.

9.3.13.2 Heat Transfer

As stated in an adiabatic process, Q = 0.

Applying an energy balance to this case (Figure 9.13b):

$$U_1 - W = U_2$$

$$W = U_1 - U_2 \tag{9.65}$$

In an adiabatic expansion, the work output will be equal to the decrease in internal energy, that is, because of the work output, the internal energy of the system will decrease by a corresponding amount.

Hence, for a perfect gas:

$$U_1 = mc_vT_1$$

$$U_2 = mc_vT_2$$

Substituting into Equation 9.65:

$$W = mc_v(T_1 - T_2) \tag{9.66}$$

Now

$$c_p - c_v = R$$

or

$$c_v = \frac{R}{\gamma - 1} \tag{9.67}$$

Substituting in Equation 9.66

$$W = \frac{mR(T_1 - T_2)}{\gamma - 1} \tag{9.68}$$

But $mRT_1 = p_1V_1$ and $mRT_2 = p_2V_2$.

Substituting in Equation 9.68, the expression for the expansion becomes

$$W = \frac{p_1V_1 - p_2V_2}{\gamma - 1} \tag{9.69}$$

EXAMPLE 9.15

A thermally insulated reciprocating compressor compresses air at 0.98 bar and 20°C to one-sixth of its original volume. Calculate the final pressure and volume of the air after compression.

If the mass of air within the cylinder is 0.05 kg, determine the required work input. For air, take $\gamma = 1.4$ and $c_v = 0.718$ kJ/kg K.

SOLUTION

From the question, $p_1 = 0.98$ bar, $T_1 = 20 + 273 = 293$ K, $V_2/V_1 = 1/6$, $m = 0.05$ kg and $c_v = 0.718$ kJ/kg.

As the cylinder is thermally insulated, it is considered that heat transfer is negligible and that the process is treated as adiabatic and considers the air as a perfect gas.

From Equation 9.61:

$$\frac{p_2}{p_1} = \left[\frac{V_1}{V_2}\right]^{\gamma}$$

Now

$$p_2 = 0.98 \times 6^{1.4}$$

$$= 12.04 \text{ bar}$$

From Equation 9.62:

$$\frac{T_2}{T_1} = \left[\frac{V_1}{V_2}\right]^{\gamma - 1}$$

$$T_2 = 293 \times 6^{0.4}$$

$$= 599.97 \text{ K} \quad \text{say } 600 \text{ K}$$

$$= 327°\text{C}$$

Rewriting Equation 9.66:

$$W = mc_v(T_1 - T_2)$$

$$W = 0.05 \times 0.718\ (600 - 293)$$

$$= 11.02 \text{ kJ}$$

9.3.14 Polytropic Process ($pV^n = C$)

In practice, it is found that many processes approximate to a reversible law in the form

$$pV^n = \text{constant} \qquad (9.70)$$

where n is a constant. Vapours and perfect gases obey this type of law closely in many non-flow processes. Such processes are internally reversible.

If a piston in a cylinder is cooled perfectly and a compression or expansion is carried out slowly, the process will be isothermal where $n = 1$.

If a compression is carried out rapidly and again the piston and cylinder are perfectly insulated, the process will be adiabatic and in this case the constant $n = \gamma$.

When a compression or expansion is carried out at a moderate speed and the piston and cylinder assembly are cooled to some degree, the process will be somewhere between the two conditions described above. This will be the general situation in many engineering applications where the index ‘n’ will take some value that is between 1 and γ dependent upon the degree of cooling.

Some examples include

Compression in a water-cooled air compressor: n = 1.1
Compression in a fan-cooled air compressor: n = 1.2
Compression in an air-cooled air compressor: n = 1.3

Figure 9.14a and b depicts the polytropic process.

Equation 9.70 applies to the states 1 and 2 as

$$p_1 V_1^n = p_2 V_2^n \tag{9.71}$$

or

$$\frac{p_2}{p_1} = \left[\frac{V_1}{V_2}\right]^n \tag{9.72}$$

For a perfect gas, the general property relationship between the two states:

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2} \tag{9.73}$$

Manipulation of Equations 9.71 and 9.72 results in the following relationship:

$$\frac{T_2}{T_1} = \left[\frac{p_2}{p_1}\right]^{\frac{n-1}{n}} = \left[\frac{V_1}{V_2}\right]^{n-1} \tag{9.74}$$

When examining Equations 9.72 and 9.74, the following conclusions can be made:

- Any increase in volume will result in a decrease in pressure.
- Any increase in volume will result in a decrease in temperature.
- Any increase in pressure will result in an increase in temperature.

9.3.14.1 Work Transfer

Referring to Figure 9.14b, there is an increase in volume during the process and as the fluid expands, the expansion work is given by

$$W = \int_1^2 p\, dV$$

$$= \int_1^2 \frac{c}{V^n} dV \quad (pV^n = c, \text{a constant})$$

$$= c\int_1^2 \frac{dV}{V^n} \tag{9.75}$$

$$W = \frac{p_1V_1 - p_2V_2}{n-1} \tag{9.76}$$

Note that after expansion, p_2 is smaller than p_1.

Following the expansion process, the work is done by the system and this is shown by the direction of the arrow that represents work 'W' as shown in Figure 9.14b.

9.3.14.2 Heat Transfer

The energy balance is applied to this case as

$$U_1 - Q_{loss} - W = U_2$$

$$Q_{loss} = (U_1 - U_2) - W \tag{9.77}$$

or

$$W = (U_1 - U_2) - Q_{loss} \tag{9.78}$$

In a polytropic expansion, the work output is reduced due to the heat losses.

Examining Equation 9.78 and Figure 9.14b, it will be noted that during the polytropic process, as the volume increases there is a corresponding reduction in pressure. For a perfect gas, Equation 9.74 also shows that a decrease in pressure will also result in a drop in the system temperature.

EXAMPLE 9.16

Following compression, the combustion gases in a petrol engine are at 35 bar and 900°C. The gases then expand through a volume ratio (V_2/V_1) of 8.5/1 and occupy 0.51×10^{-3} m^3 after expansion. The polytropic expansion index n = 1.15 when the engine is air cooled. Calculate the temperature and pressure of the gas after expansion and establish what the work output will be?

Solution

From the question, $p_1 = 30$ bar, $t_1 = 900$°C ($T_1 = 900 + 273 = 1173$ K), $V_2 = 0.51 \times 10^{-3}$ m^3, n = 1.15 and $V_2/V_1 = 8.5$.

Treating the air as a perfect gas, for a polytropic process the property relationship is given by Equation 9.74

$$\frac{T_2}{T_1} = \left[\frac{V_1}{V_2}\right]^{n-1}$$

$$= 1173 \times \left[\frac{1}{8.5}\right]^{1.15-1}$$

$$= 850.9\ \text{K}$$

$$= 577.9°\text{C}$$

From Equation 9.72:

$$p_2 = p_1\left[\frac{V_1}{V_2}\right]^n$$

$$= 35 \times \left[\frac{1}{8.5}\right]^{1.15}$$

$$= 2.99\ \text{bar}$$

and

$$\frac{V_2}{V_1} = 8.5$$

Then

$$V_1 = \frac{0.510 \times 10^{-3}}{8.5}$$

$$= 60 \times 10^{-6}\ \text{m}^3$$

Equation 9.76 gives the work output during the polytropic expansion:

$$W = \frac{p_1V_1 - p_2V_2}{n - 1}$$

$$= \frac{(35 \times 10^5)(60 \times 10^{-6}) - (2.99 \times 10^5)(0.510 \times 10^{-3})}{1.15 - 1}$$

$$= 529.6\ \text{J}$$

$$= 0.530\ \text{kJ (Ans)}$$

9.3.15 Constant Volume Process

In certain chemical processes, fluids are held in fixed volume rigid-walled vessels whilst the fluid or gas is either heated or cooled as shown in Figure 9.15a and b. In this case, the process is considered a constant volume process as the vessel has a fixed volume.

The general property relation between the initial and final states of a perfect gas is applied as

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

As the volume remains constant during the process, $V_2 = V_1$ and the above equation then reduces to

$$\frac{p_1}{T_1} = \frac{p_2}{T_2} \tag{9.79}$$

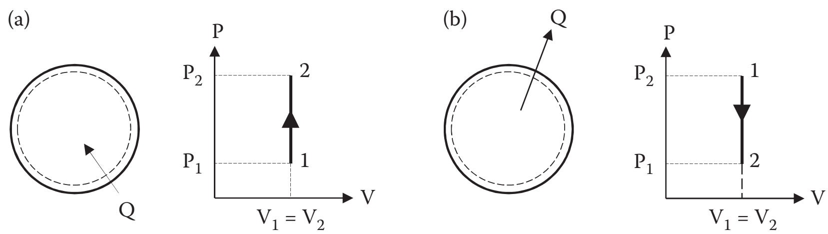


FIGURE 9.15 Constant volume process ($V_1 = V_2$). (a) Heating and (b) cooling.

or

$$\frac{T_2}{T_1} = \frac{p_2}{P_1} \tag{9.80}$$

From this equation, it is clearly seen that an increase in temperature will result in a corresponding increase in pressure. Hence, the temperature is proportional to the pressure.

9.3.15.1 Work Transfer

Work transfer (p dV) will be zero as the change in volume (dV) during the process will also be zero. Some process vessels (reactors) usually have a stirrer or paddlewheel installed to assist in mixing the fluid, in which case some work will be transferred.

9.3.15.2 Heat Transfer

Applying the non-flow energy equation from Equation 9.55

$$Q - W = U_2 - U_1$$

This will give

$$Q - 0 = U_2 - U_1$$

that is,

$$Q = U_2 - U_1 \tag{9.81}$$

Note: This result is important and shows that the net amount of heat energy supplied to or taken from the fluid during a constant volume process is equal to the change in the internal energy of the fluid.

EXAMPLE 9.17

During a constant volume process, the specific internal energy of a fluid is increased from 120 kJ/kg to 180 kJ/kg. Calculate the amount of heat energy supplied to 2 kg of fluid to increase the internal energy.

Solution

From the non-flow energy equation:

$$Q - W = U_2 - U_1$$

For a constant volume process:

$$W = 0$$

Therefore, the equation becomes

$$Q = U_2 - U_1$$
$$= (180 - 120)\ \text{kJ/kg}$$
$$= 60\ \text{kJ/kg}$$

For a mass of 2 kg of fluid:

$$Q = 60 \times 2$$
$$= 120\ \text{kJ/kg}$$

that is, 120 kJ/kg of heat energy will be required.

9.3.16 Constant Pressure Process

Intensifiers are used in hydraulic and gas supply systems to maintain a constant pressure within the system, even though the flow may be varying due to usage. These consist of having a cylinder fitted with a piston that has a constant load applied to it as depicted in Figure 9.16a and b. This is an example of a constant pressure process.

The general property relation between the initial and final states of a perfect gas is applied as

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$

When the pressure remains constant during the process, $p_2 = p_1$, the above relation then becomes

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$

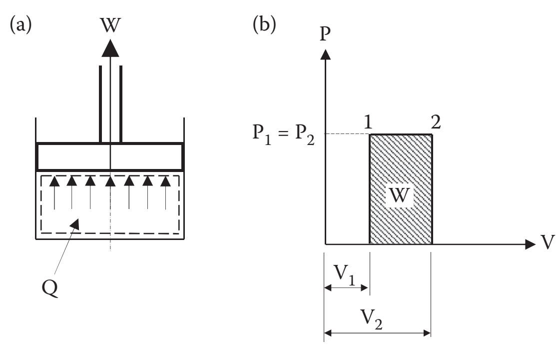


FIGURE 9.16 (a) and (b): Constant pressure process.

or

$$\frac{T_2}{T_1} = \frac{V_2}{V_1} \quad (9.82)$$

It is obvious from this equation that any increase in volume will result in a corresponding increase in temperature in the fluid. Therefore, the temperature is proportional to the volume.

9.3.16.1 Work Transfer

Referring to the p–V diagram in Figure 9.16b, it will be noted that the increase in volume expands during the process. This expansion work is given by (similar to Equation 9.63)

$$W = \int_1^2 p\,dV$$

$$= p\int_1^2 dV \quad \text{(p is constant)}$$

$$= p(V_2 - V_1) \quad (9.83)$$

On a p–V diagram, the area under the process line represents the amount of work transfer. From Figure 9.16b

$$\begin{aligned} W &= \text{Area of the shaded rectangle} \\ &= \text{Height} \times \text{width} \\ &= p(V_2 - V_1) \quad \text{(It will be noted that this expression is identical to Equation 9.83.)} \end{aligned}$$

9.3.16.2 Heat Transfer

Again applying the non-flow energy equation

$$Q - W = U_2 - U_1$$

or

$$Q = (U_2 - U_1) + W \quad (9.84)$$

Part of the heat supplied is converted into work energy and the remainder is utilised in increasing the internal energy of the system.

Now, substituting for 'W' in Equation 9.84:

$$\begin{aligned} Q &= (U_2 - U_1) + p(V_2 - V_1) \\ &= U_2 - U_1 + p_2V_2 - p_1V_1 \quad (\text{since } p_2 = p_1) \\ &= (U_2 + p_2V_2) - (U_1 + p_1V_1) \end{aligned}$$

Now

$$H = U + pV$$

Hence,

$$Q = H_2 - H_1 \tag{9.85}$$

Referring to Figure 9.16b, it will be noted that heating will increase the volume, that is, the fluid will expand. For a perfect gas, the equation suggests that an increase in volume will result in a corresponding increase in temperature.

EXAMPLE 9.18

A cylinder contains a fluid with a volume of 0.1 m^3 at a constant pressure of 7 bar and having a specific enthalpy of 210 kJ/kg. The volume expands to 0.2 m^3 following the application of heat energy to the fluid and the specific enthalpy increases to 280 kJ/kg.

Determine:

1. The quantity of heat energy supplied to the fluid.
2. The change in the internal energy of the fluid.

Solution

From the question, $p = 7.0$ bar, $V_1 = 0.1\ m^3$ and $V_2 = 0.2\ m^3$.

1. Heat energy supplied = change in enthalpy of the fluid:

$$\begin{aligned} Q &= H_2 - H_1 \\ &= m(h_2 - h_1) \\ &= 2.25(280 - 210) \\ &= 157.5\ \text{kJ} \end{aligned}$$

2. For a constant pressure process:

$$\begin{aligned} W &= P(V_2 - V_1) \\ &= 7 \times 10^5 \times (0.2 - 0.1) \\ &= 7 \times 10^4\ \text{J} \\ &= 70\ \text{kJ} \end{aligned}$$

Applying the non-flow energy equation

$$Q - W = U_2 - U_1$$

gives

$$\begin{aligned} U_2 - U_1 &= 157.5 - 70 \\ &= 87.5\ \text{kJ} \end{aligned}$$

10 Fluid Mechanics

10.1 FLUID PROPERTIES

The term fluid covers both liquids and gasses. It should be noted that there is a degree of commonality in Chapter 9, where a number of terms used are also used in fluid mechanics. Table 10.1 lists some of these terms used together with their units.

A liquid is hard to compress and takes the shape of the container it is in. It has a fixed volume and an upper-level surface. However, although liquids are generally considered to be incompressible, they are compressible only when they are highly pressurised.

Gasses are very easy to compress and expand to fill their container and unlike liquids, they do not have a free surface.

The important characteristics of fluids from the point of view of fluid mechanics include density, pressure, viscosity and compressibility.

10.1.1 Density

There are three forms of density and a distinction must be carefully made between them.

1. *Mass density (ρ).* This is the mass per unit volume with the unit of kilogram per cubic meter. The density of gas changes according to its pressure, but that of a liquid may be considered constant unless the relevant pressures are very high.
2. *Specific weight (⊠).* In fluid mechanics, specific weight represents the force exerted by gravity on a unit volume of a fluid. For this reason, the unit is expressed as force per unit volume (N/m^3). Specific weight can be used as a characteristic property of a fluid.
3. *Specific gravity (sg) or relative density (RD).* This is a dimensionless quantity being the ratio of the density (mass of a unit volume) of a substance to the density of a given reference material. Sg usually means relative density with respect to water. The term 'relative density' is often preferred in modern scientific usage.

The sg for liquids and solids is generally identified as the ratio of the material density relative to water under the same conditions, say 4°C and 1.013 bar (ambient pressure). For gasses, the sg is the ratio of the gas density to the density of air, both under the same conditions as identified above.

For a liquid:

$$RD = \frac{\rho}{\rho_w} \qquad (10.1)$$

For a gas:

$$D = \frac{\rho}{\rho_a} \qquad (10.2)$$

TABLE 10.1
Symbols Used in Fluid Mechanics

Symbol	Description	Units	Symbol	Description	Units
a	Acceleration	m/s^2	R	Gas constant	J/(kg · K)
A	Area	m^2	R_o	Universal gas constant	J/(kg · mol · K)
F	Force	N	ρ	Fluid density	$kg \cdot m^3$
g	Acceleration due to gravity	m/s^2	ρ	Density	kg/m^3
h	Fluid head	m	ρ_r	Density	kg/m^3
K	Bulk modulus	MPa	s	Specific volume	m^3/kg
m	Mass	kg	u	Fluid velocity	m/s
M	Molecular weight		v	Fluid velocity	m/s
p	Fluid pressure	N/m^2	x	Depth of centroid	m
P_{abs}	Absolute pressure	N/m^2	β	Compressibility	1/MPa
P_{gauge}	Gauge pressure	N/m^2	θ	Slope	Radians
P_{atm}	Atmospheric pressure	N/m^2	τ	Shear stress	N/m^2
P_s	Surface pressure	N/m^2	μ	Viscosity	Pa · s
Q	Volumetric flow	m^3/s	ν	Kinematic viscosity	m^2/s
q	Heat transfer/unit mass	J/kg	υ	Specific volume	m^3/kg
			γ	ratio of specific heats	

The density of gasses and vapours is greatly affected by pressure. For 'perfect gasses', the density can be calculated from the formula:

$$\rho = \frac{P}{RT} \tag{10.3}$$

$$R = \frac{R_o}{M} \tag{10.4}$$

'R_o' is the universal gas constant = 8314 J/kg K and 'M' is the molecular weight of the material. Hence,

$$R = \frac{8314}{M}\frac{J}{kg\ K} \tag{10.5}$$

The reciprocal of density, that is, the volume per unit mass is called the specific volume (υ)

$$\upsilon = \frac{1}{\rho} \tag{10.6}$$

The dimensionless unit for density is ML^{-3} and the corresponding unit for specific volume is $M^{-1}L^3$ (see Section 10.5).

10.1.2 Pressure

On planet Earth, a fluid is always subject to pressure and is the force per unit area at a point. The absence of pressure can only occur in a complete vacuum. A complete vacuum is really a theoretical concept as even in deep space, there is a partial pressure.

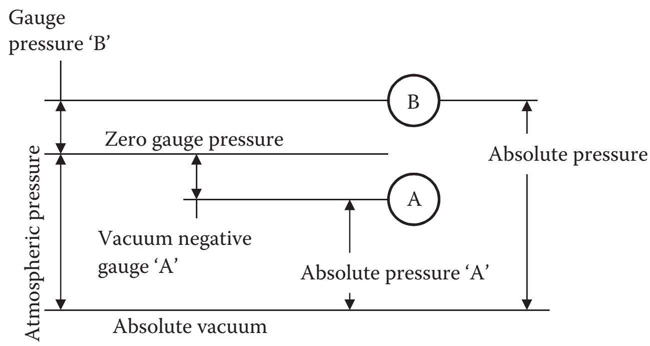


FIGURE 10.1 Atmospheric scale.

The normal pressure occurring on the surface of the Earth is called the 'atmospheric pressure'. Pressure is measured in two ways:

- Absolute pressure, where the pressure is measured relative to that of a perfect vacuum.
- Gauge pressure, where the pressure is measured relative to the local atmospheric pressure. These measured pressures are referred to as gauge pressures.

Figure 10.1 shows the relationship between the gauge and the atmospheric pressure. The figure shows two measurements:

1. A pressure less than atmospheric pressure
2. A pressure greater than atmospheric pressure

The SI unit for pressure is Pascal (Pa) with the unit of N/m^2.

10.1.3 Static Pressure and Head

When considering fluid pressure, it has been found more convenient in hydrostatics and fluid dynamics to use a pressure head as a means of measuring pressures. Considering Figure 10.2, a volume of fluid in an open container is subject to atmospheric pressure acting on the surface of the fluid. Consider a vertical tube that is connected to the bottom of the container and terminates at a

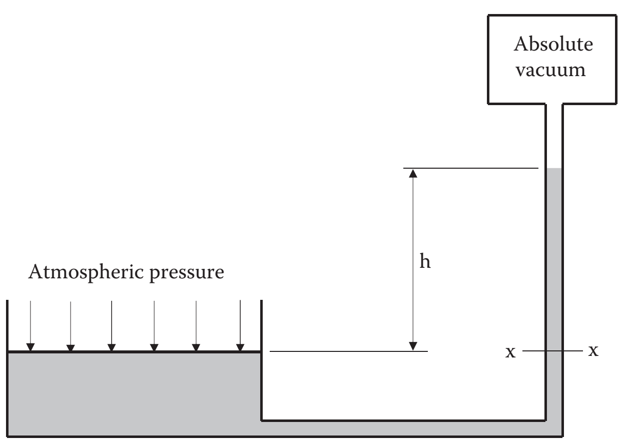


FIGURE 10.2 Atmospheric pressure.

distance above the surface of the fluid into a vessel held at a pressure of absolute vacuum. It will be found that the fluid will be forced up the tube to a point where gravity acting on the fluid in the tube balances the force due to the pressure at the bottom of the head of the fluid. Assuming that the area of the tube is A_t and the density of the fluid is ρ, the pressure at the top of the tube is zero. The force of the fluid acting on the section 'x:x' as shown in the figure will be

$$F_{xx} = 0 + hA\rho g \tag{10.7}$$

The pressure at 'x:x' is equal to

$$P_{xx} = \frac{0 + hA\rho g}{h\rho g}$$

For a fluid having a known fixed density, the height 'h' can be conveniently used to identify the pressure. For water, the atmospheric pressure is approximately 10.5 m. In practice, water will vaporise into the vacuum at the top of the tube, thereby reducing the column height by approximately 180 mm.

Therefore, mercury is used for measuring pressure. With an atmospheric pressure of 1 bar, this will support a column height of 750.06 mm of mercury.

Mercury is used as it has a low-vapour pressure and the vacuum will only be reduced by approximately 0.16 Pa. This is very small compared to an atmospheric pressure of 10^5 Pa (1 bar).

From this description, it is clear that gauge and vacuum pressures can be easily determined using this simple method. As a point of interest, barometers measure the local atmospheric pressure readings in millimeters of mercury (Hg).

10.1.4 Viscosity

A fluid at rest cannot resist any shearing forces, but once the fluid is in motion, shearing forces are then set up between layers of the fluid with them moving at different velocities. The viscosity of a fluid is a measure of the ability of the fluid to resist these shearing forces.

Perfect fluids cannot, in theory, transmit shear stresses. All real fluids will resist shear flow to some degree or another. The viscosity of the fluid defines the degree of resistance to flow it possesses. Consider Figure 10.3a that shows a concentric tube located on a shaft and separated from the shaft by a fluid. As the shaft rotates, there will be a tendency for the tube to rotate at an angular velocity (ω).

The velocity distribution is shown in Figure 10.3b. The torque required to rotate the tube is an indication of the viscosity of the fluid.

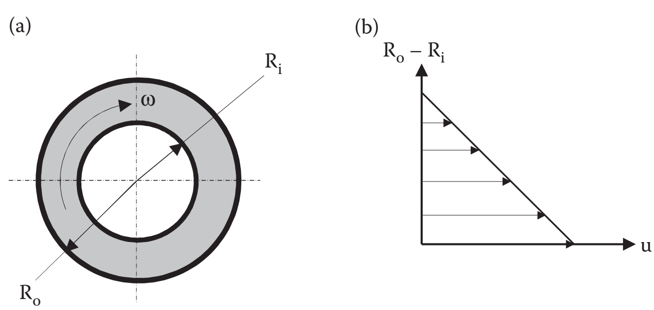


FIGURE 10.3 (a) and (b): Viscosity of a fluid.

10.1.4.1 Coefficient of Dynamic Viscosity

The coefficient of dynamic viscosity (η) is defined as the shear force per unit area that is required to drag one layer of a fluid with a unit velocity past another layer a unit distance away from it in the fluid. The SI unit of measurement for dynamic viscosity is

$$\text{Coefficient of dynamic viscosity} = \frac{\text{N} \cdot \text{s}}{\text{m}^2} \ \text{or} \ \frac{\text{kg}}{\text{m} \cdot \text{s}} \ \text{or} \ \text{Pa} \cdot \text{s}$$

The old unit for dynamic viscosity (η) was dyn · s/cm^2 and this was called a 'poise' (after Poiseuille) and the SI unit is related to the Poise:

$$10 \text{ Poise} = 1\frac{\text{Ns}}{\text{m}^2} \quad \text{(this is not considered an acceptable multiple)}$$

Since

$$1 \text{ centipoise (1 cP)} = 0.001\frac{\text{N} \cdot \text{s}}{\text{m}^2} \quad \text{cP is an accepted SI unit}$$

10.1.4.2 Kinematic Viscosity

Kinematic viscosity (v) is the ratio of the dynamic viscosity to the mass density:

$$v = \frac{\eta}{\rho} \tag{10.8}$$

The base unit is m^2/s. The previous metric unit was cm^2/s and this was called the 'Stoke' after a British scientist. The SI unit is related to the Stoke.

$$1 \text{ Stoke (St)} = 0.0001\frac{\text{m}^2}{\text{s}} \quad \text{(this is not considered acceptable as an SI unit)}$$

Since

$$1 \text{ CentiStoke (cSt)} = 0.000001\frac{\text{m}^2}{\text{s}} \quad \text{and this is an acceptable multiple}$$

$$1 \text{ cSt} = 0.00000\frac{\text{m}^2}{\text{s}} = 1\frac{\text{mm}^2}{\text{s}}$$

10.1.4.3 Other Units

Other units of viscosity will be encountered by the way they were originally derived. As an example, Redwood seconds came from the name of the Redwood viscometer. Other units include Engler's degrees, Society of Automotive Engineers (SAE) numbers and so on. Conversion charts are available to enable conversions into useable engineering or SI units.

10.1.5 Compressibility

For liquids, the relationship between the change in pressure and the change in volume is given by the bulk modulus 'K'

$$\text{Bulk modulus} = \frac{\text{Change in pressure intensity}}{\text{Volumetric strain}} \tag{10.9}$$

$$= \frac{\text{Change in pressure intensity}}{(\text{Change in volume/original volume})} \qquad (10.10)$$

The relationship between pressure and volume for a gas can be derived from the gas laws.

For all perfect gases:

$$pV = RT \qquad (10.11)$$

where

p represents absolute pressure
V represents specific volume = 1/w
T represents absolute temperature
R represents gas constant

If any change occurs isothermally (at constant temperature):

$$pV = \text{constant} \qquad (10.12)$$

If any change occurs adiabatically (no heat gain or loss):

$$pV^{\gamma} = \text{constant} \qquad (10.13)$$

where γ represents the ratio of the specific heat at constant pressure to the specific heat at constant volume.

10.2 FLUID FLOW

When a fluid is at rest, there is no shear force developed but when the fluid is in motion, shear forces set up due to viscosity and turbulence, opposing the motion and producing frictional effects.

This section will study the patterns of flow, both inside a conduit and outside it.

10.2.1 Patterns of Flow

A fluid may be considered as consisting of a large number of individual particles moving in the general direction of flow but usually, are not parallel with each other. The velocity of any particle is a vector quantity having a magnitude and direction that varies from moment to moment. The path followed by a particle is called a 'streamline' or 'path line'.

When considering the flow of a large body of fluid, it is sometimes convenient to consider a small section. If streamlines are drawn through every point on the circumference of a small area, a 'stream tube' is formed (see Figure 10.4). Fluid particles can only flow along a streamline and particles cannot flow across the streamline; the fluid inside a stream tube can only enter or leave at its ends.

10.2.2 Types of Flow

There are two types of flow.

10.2.2.1 Internal Flow

Internal flow is within the boundary walls and these include all types of flow within pipes, channels, airflow within ducts and so on.

10.2.2.2 External Flow

External flow covers the flow that is outside a boundary, body or conduit. Examples of these types of flow include immersed bodies, airflow around buildings and flow over aircraft wings and suspension bridges and around automobile vehicles.

Two distinct types of flow can occur; these are as follows.

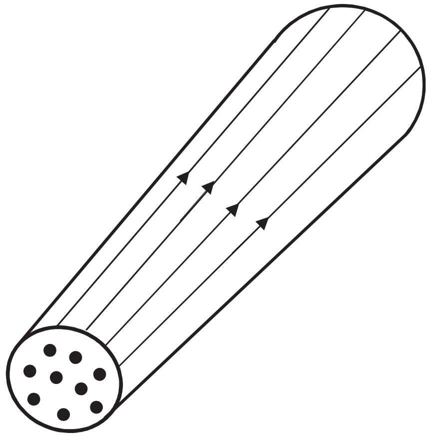


FIGURE 10.4 Stream tube.

10.2.2.3 Laminar Flow

This type of flow is also known as viscous flow or streamline in which the particles move in an orderly manner and retain the same relative positions in successive cross sections.

10.2.2.4 Turbulent Flow

In this type of flow, it may be visualised that the particles of fluid move in a disorderly manner if they occupy different relative positions in successive cross sections.

10.2.3 Laminar Flow

Following a study of laminar flow, Osborne Reynolds concluded that this type of flow can be defined by the velocity, density and viscosity of the fluid together with the dimensional size of the conduit and depends on the value of

$$\frac{\rho v d}{\eta} \tag{10.14}$$

This relationship is known as 'Reynolds number', in which

ρ represents the mass density of the fluid
v represents velocity
d represents a typical dimension for the size of the pipe
η represents the viscosity of the fluid

When the Reynolds number is <2100 approximately, the flow will always be viscous. If the relevant Reynolds number is >4000, the flow will be turbulent. When the flow in the transition region is termed 'critical', it may be either laminar or turbulent or a mixture of both.

EXAMPLE 10.1

A fluid with a density of 860 kg/m^3 has a kinematic viscosity of 40 cSt. Determine the critical velocity when it is flowing through a pipe having a bore of 50 mm.

Solution

$$R_e = \frac{u_m d}{v}$$

$$u_m = \frac{R_e\, v}{d}$$

$$= \frac{2100 \times 40 \times 10^{-6}}{0.05}$$

$$= 1.68 \text{ m/s}$$

10.2.4 Derivation of Poiseuille's Equation for Laminar Flow

The original derivation for laminar flow was undertaken by Poiseuille and relates pressure loss in a pipe to the velocity and viscosity. The equation is the basis for the measurement of viscosity; hence, his name has been used for the measurement of viscosity.

Consider a pipe having a laminar flow in it. A stream tube of length ⊠L, a radius 'r' and thickness dr is shown in Figure 10.5.

Let 'y' be the distance from the pipe wall:

$$y = R - r \tag{10.15}$$

$$dy = -dr \quad \frac{du}{dy} = -\frac{du}{dr} \tag{10.16}$$

The shear stress on the outside of the stream tube is 'τ'. The force (F_s) acting from right to left is due to the shear stress and is obtained by multiplying 'τ' by the surface area.

Hence,

$$F_s = \tau \times 2\pi r \Delta L \tag{10.17}$$

For a Newtonian fluid:

$$\tau = \mu \frac{du}{dy} = -\mu \frac{du}{dr} \tag{10.18}$$

Substituting for 'τ', the following equation is obtained:

$$F_s = -2\pi r \Delta L \mu \frac{du}{dr} \tag{10.19}$$

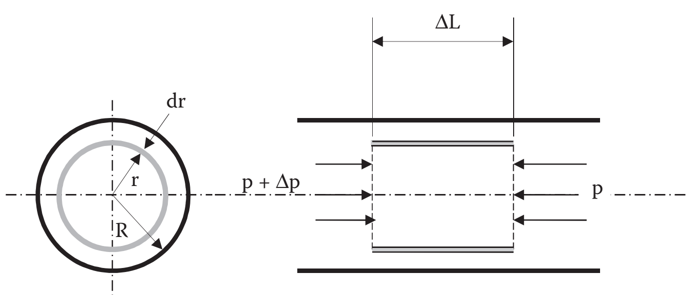


FIGURE 10.5 Derivation for laminar flow.

The pressure difference between the left end and the right end of the section is Δp. The force due to F_p is Δp × circular area of radius ‘r’.

$$F_p = \Delta p \times pr^2$$

Equating forces:

$$-2\pi r\mu\Delta L\frac{du}{dr} = \Delta p\pi r^2$$

$$du = \frac{\Delta p}{2\mu\Delta L}rdr \tag{10.20}$$

Integrate Equation 10.20 to obtain the velocity of the streamline at any radius ‘r’ between the limits u = 0 when r = R and u = u when r = r.

$$\int_0^u du = -\frac{\Delta p}{2\mu\Delta L}\int_R^r rdr$$

$$u = -\frac{\Delta p}{4\mu\Delta L}(r^2 - R^2)$$

$$u = \frac{\Delta p}{4\mu L}(r^2 - R^2) \tag{10.21}$$

It is recognised that this is the equation for a parabola; so, when the equation is plotted to show the boundary layer, it extends from zero to a maximum value at the centre (Figure 10.6).

For the maximum velocity, substitute r = 0:

$$u_1 = \frac{\Delta pR^2}{4\mu\Delta L} \tag{10.22}$$

The average height of a parabola is the maximum value/2; so, the average velocity

$$u_m = \frac{\Delta pR^2}{8\mu\Delta L} \tag{10.23}$$

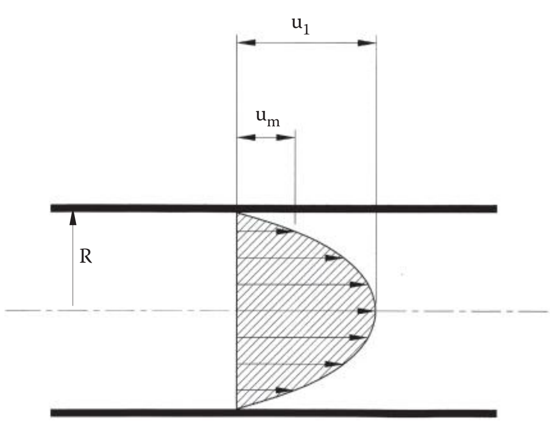


FIGURE 10.6 Laminar flow in a pipe.

It may be required to calculate the pressure drop in terms of diameter 'd'. Therefore, substitute R = d/2 and rearrange the equation:

$$\Delta p = \frac{32\mu\Delta L\mu_m}{d^2} \tag{10.24}$$

The volumetric flow rate is

Average velocity × cross-sectional area

Hence,

$$Q = \frac{\pi R^2 \Delta p R^2}{8\mu\Delta L}$$

$$= \frac{\pi R^4 \Delta p}{8\mu\Delta L}$$

$$= \frac{\pi d^4 \Delta p}{128\mu\Delta L} \tag{10.25}$$

This equation can be rearranged for the pressure head; the friction head for a length 'L' is derived from

$$h_f = \frac{\Delta p}{\rho g} \tag{10.26}$$

Hence,

$$h_f = \frac{32\mu L\mu_m}{\rho g d^2} \tag{10.27}$$

This is Poiseuille's equation and applies only to laminar flow.

EXAMPLE 10.2

A capillary tube is 30 mm long and has a 1.0 mm bore. To obtain a flow rate of 8 mm^3/s requires a head of 30 mm. The density of the fluid is 800 kg/m^3.

Calculate the dynamic and kinematic viscosity of the fluid.

SOLUTION

Rearranging Poiseuille's equation:

$$\mu = \frac{hf\rho g d^2}{32Lu_m}$$

Now

$$A = \frac{\pi d^2}{4}$$

$$= \frac{\pi \times 1.0^2}{4}$$

$$= 0.7854 \text{ mm}^2$$

$$u_m = \frac{Q}{A}$$
$$= \frac{8}{0.7854}$$
$$= 10.18 \text{ mm/s}$$

Substituting values:

$$\mu = \frac{0.03 \times 800 \times 9.81 \times 0.001^2}{32 \times 0.03 \times 0.01018}$$
$$= 0.02409 \text{ N} \cdot \text{s/m} \quad \text{or}$$
$$= 241 \text{ cP}$$

$$v = \frac{\mu}{\rho}$$
$$= \frac{0.02409}{800}$$
$$= 30.114 \times 10^{-6} \text{ m}^2/\text{s} \quad \text{or}$$
$$= 30.11 \text{ cSt}$$

EXAMPLE 10.3

A fluid is flowing through a pipe of 150 mm bore having a Reynolds number of 250. The dynamic viscosity is 0.018 $N \cdot s/m^2$ and the density is 900 kg/m^3.

Calculate the pressure drop per metre length together with the velocity and radius at which it occurs.

Solution

$$R_e = \frac{\rho \mu_m d}{\mu}$$

$$\mu_m = \frac{R_e \cdot \mu}{\rho d}$$

$$\mu_m = \frac{(250 \times 0.018)}{(900 \times 0.15)}$$
$$= 0.0333 \text{ m/s}$$

$$\Delta p = \frac{32 \mu L \mu_m}{d^2}$$
$$= \frac{32 \times 0.018 \times 1.0 \times 0.05}{0.15^2}$$
$$= 1.28 \text{ Pa}$$

$$u = \left(\frac{\Delta p}{4 L \mu}\right)(R^2 - r^2)$$

The above equation is rearranged to solve for 'r':

$$r = \sqrt{R^2 - \frac{4L\mu u}{\Delta p}}$$
$$= 0.0692\,\text{m}$$
$$r = 69.2\,\text{mm}$$

10.2.5 Turbulent Flow

In the previous section, it was shown that Poiseuille's equation applies only to laminar flow. At some critical velocity, the flow will begin to become turbulent forming eddies and showing other chaotic behaviour that do not contribute to the volumetric flow rate. This turbulence increases the resistance dramatically so that large increases in pressure will be required to further increase the volume flow rate.

In a circular pipe, turbulence is generated by the friction existing on the internal surface of the pipe. If the internal surface was completely smooth without any surface roughness, then the flow would remain laminar at higher Reynolds numbers into the critical zone. Where there is any surface roughness, this will initiate turbulence at lower Reynolds numbers. Consider Figure 10.7 that shows the internal surface roughness of a pipe. Relative roughness is expressed as D/ε.

That is, less rough pipes will have a high D/ε whereas more rough pipes will have lower D/ε.

Table 10.2 gives some comparative roughness values for materials commonly used in fluid transport.

To begin the discussion of turbulent flow in circular pipes, it is necessary to define the friction coefficient. It is defined as follows:

$$C_f = \frac{\text{wall shear stress}}{\text{dynamic pressure}}$$
$$= \frac{2D\Delta p}{4L\rho\mu_m^2} \tag{10.28}$$

Rearranging Equation 10.28 to equate to Δp:

$$\Delta p = \frac{4C_f L\rho\mu_m^2}{2D} \tag{10.29}$$

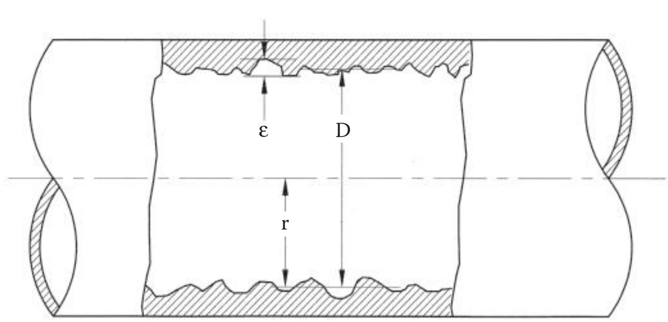


FIGURE 10.7 Surface roughness in a pipe.

TABLE 10.2
Comparative Roughness Values for a Range of Materials

Material	Surface Roughness ε (m)
Plastic	3.0×10^{-7}
Steel plain	4.6×10^{-5}
Steel galvanised	1.5×10^{-4}
Concrete	1.2×10^{-4}

This expression is frequently used to state the friction head h_f.

$$h_f = \frac{\Delta p}{\rho g} = \frac{4C_f L u_m^2}{2gD} \quad (10.30)$$

This expression is known as the 'Darcy formula' and in the case of laminar flow, Darcy's formula must give the same result as Poiseuille's equation.

Hence, equating the two expressions:

$$\frac{4C_f L u_m^2}{2gD} = \frac{32\mu L u_m}{\rho g D^2} \quad (10.31)$$

Hence

$$C_f = \frac{16\mu}{\rho u_m D} = \frac{16}{R_n} \quad (10.32)$$

This gives the same result as for laminar flow.

In pipes where the Reynolds number exceeds 3000, it can be safely assumed that turbulent flow will exist.

10.2.6 Fluid Resistance

An alternative approach to solving problems that involve losses is to use fluid resistance. The above equations can be expressed in terms of the flow rate 'Q' by substituting u = Q/A.

From Equation 10.30:

$$h_f = \frac{4C_f L u_m^2}{2gD} \quad (10.33)$$

$$= \frac{4C_f L Q^2}{2gDA^2} \quad (10.34)$$

Substituting $A = \pi D^2/4$, the following equation is derived:

$$h_f = \frac{32 C_f L Q^2}{g\pi^2 D^5}$$

$$= RQ^2 \quad (10.35)$$

where R is the fluid resistance; hence

$$R = \frac{32C_f L}{g\pi^2 D^5} \tag{10.36}$$

Note: This equation contains the friction coefficient (C_f) and this will vary with changes in velocity and surface roughness; therefore, 'R' should be used with caution.

10.2.7 Moody's Diagram

An American engineer L. F. Moody (1944) conducted a series of experiments that resulted in a chart known as the Moody chart (see Figure 10.8). This shows a graph in non-dimensional form that relates friction factor, Reynolds number and the relative roughness in circular pipes.

The diagram shows that the friction coefficient decreases with the Reynolds number but at a certain point, it becomes constant. When this point is reached, the flow is said to be a fully developed turbulent. This point occurs at lower Reynolds numbers for rougher pipes.

A formula that gives an estimated value for any surface roughness was proposed by Professor Haaland of the Norwegian Institute of Technologies in 1984.

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{R_n} + \left(\frac{e}{3.71}\right)^{1.11}\right\} \tag{10.37}$$

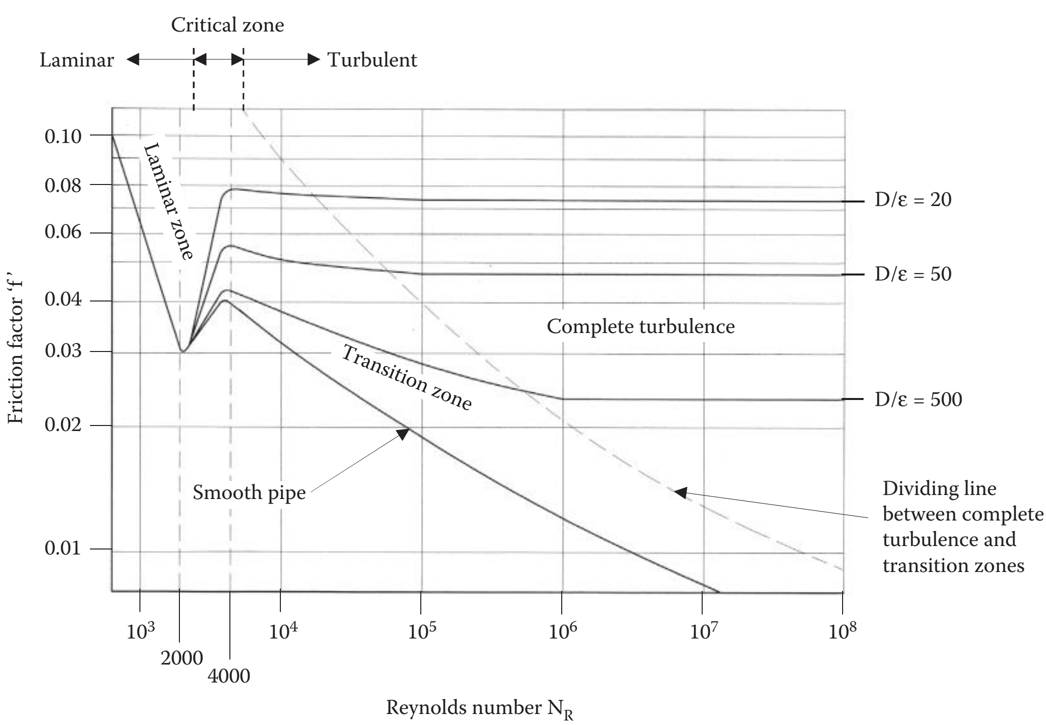


FIGURE 10.8 The Moody diagram.

EXAMPLE 10.4

A circular pipe with a 100-mm-diameter bore, having a mean surface roughness of 0.06 mm, has a fluid flowing through it with a Reynolds number of 20,000.

Calculate the friction coefficient.

Solution

The mean surface roughness:

$$\varepsilon = k/d$$
$$= \frac{0.06}{100}$$
$$= 0.006$$

From the Moody diagram (Figure 10.8), locate the line for ε = k/d.

Trace this line till it meets the vertical line at $R_n = 20{,}000$. Read off the value for C_f on the left-hand axis.

The answer will be $C_f = 0.0067$.

This answer can be checked using the Haaland formula:

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{R_n} + \left(\frac{e}{3.71}\right)^{1.11}\right\}$$
$$= -3.6\log_{10}\left\{\frac{6.9}{20{,}000} + \left(\frac{0.0006}{3.71}\right)^{1.11}\right\}$$
$$\frac{1}{\sqrt{C_f}} = 12.206$$
$$C_f = 0.0067 \text{ (Ans)}$$

EXAMPLE 10.5

A fluid flows through a pipe with a bore of 100 mm diameter and has a mean velocity of 6.0 m/s. The mean surface roughness is 0.03 mm and the length of the pipe is 100 m.

If the dynamic viscosity is 0.005 N · s/m² and the density of the fluid is 900 kg/m³, calculate the pressure loss.

From the question:

$\rho = 900$ kg/m³
$u = 6.0$ m/s
$d = 0.10$ m
$\eta = 0.005$ N · s/m²
$k = 30 \times 10^{-6}$ m
$L = 100$ m

$$R_n = \frac{\rho u d}{\eta}$$
$$= \frac{900 \times 6.0 \times 0.1}{0.005}$$
$$= 108{,}000$$

$$\begin{aligned}\varepsilon &= \frac{k}{d}\\ &= \frac{30 \times 10^{-6}}{0.1}\\ &= 0.0003\end{aligned}$$

Using Haaland's formula and checking with Moody's chart:

$$\begin{aligned}\frac{1}{\sqrt{C_f}} &= -3.6\log_{10}\left\{\frac{6.9}{R_n} + \left(\frac{\varepsilon}{3.71}\right)^{1.11}\right\}\\ &= -3.6\log_{10}\left\{\frac{6.9}{108{,}000} + \left(\frac{0.0003}{3.71}\right)^{1.11}\right\}\\ &= 14.521\\ C_f &= 0.00474\end{aligned}$$

$$\begin{aligned}h_f &= \frac{4 \cdot C_f \cdot L \cdot u^2}{2 \cdot d \cdot g}\\ &= \frac{4 \times 0.00474 \times 100 \times 6.0^2}{2 \times 0.10 \times 9.81}\\ &= 34.821\,\text{m}\end{aligned}$$

$$\begin{aligned}\Delta p &= \rho \cdot g \cdot h_f\\ &= 900 \times 9.81 \times 34.821\end{aligned}$$

Pressure drop due to the length of the pipe:

$$= 307.326 \text{ kPa (Ans)}$$

10.3 CONTINUITY EQUATION

The continuity equation is a mathematical version of the principle of the conservation of mass as applied to fluid flow. There are two laws that apply:

- The law of conservation of mass (continuity equation)
- The law of conservation of energy (Bernoulli's equation)

10.3.1 Conservation of Mass

Consider a fluid flowing at a constant rate in a pipe or conduit; the mass flow rate must be the same in all sections along the length. Figure 10.9 shows the length of piping connected to a pump filling a storage tank.

The mass flow rate in any section will be

$$m = \rho A u_m \tag{10.38}$$

where

ρ represents fluid density (kg/m^3)
A represents cross-sectional area (m^2)
u_m represents mean fluid velocity (m/s)

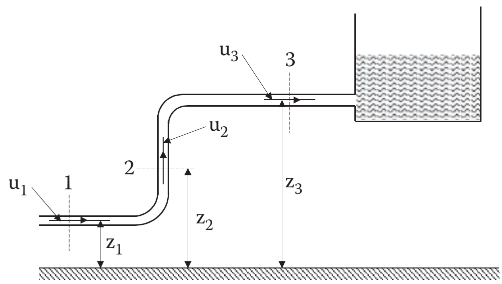


FIGURE 10.9 Conservation of mass.

For the system shown in Figure 10.9, the mass fluid flow rate must be the same in sections (1), (2), and (3); hence

$$\rho_1 A_1 u_1 = \rho_2 A_2 u_2 = \rho_3 A_3 u_3 \quad (10.39)$$

When the liquid is a fluid, the density will be equal in all sections; therefore, it cancels out, so:

$$A_1 u_1 = A_2 u_2 = A_3 u_3 = \text{constant} \quad (10.40)$$

10.3.2 Conservation of Energy

Forms of energy.

10.3.2.1 Flow Energy

This is the energy a fluid possesses due to its pressure.

$$\text{Flow energy (FE)} = p \cdot Q \text{ (joules)}$$

p represents the pressure in the fluid (Pascal)
Q represents volumetric flow rate (m^3)

10.3.2.2 Potential Energy

The energy a fluid possesses due to its height relative to a datum level.

$$\text{Potential energy (PE)} = mgz \text{ (joules)} \quad (10.41)$$

where
m represents mass (kg)
g represents gravitational constant (m/s^2)
z represents the height above the datum (m)

10.3.2.3 Kinetic Energy

This is the energy a fluid possesses due to its velocity.

$$\text{Kinetic energy (KE)} = \frac{1}{2}\mu_m^2 \text{ (joules)} \quad (10.42)$$

where ⊠ represents the mean velocity (m/s).

10.3.2.4 Specific Energy

Specific energy is the energy per kilogram.

There are three forms of specific energy:

$$1.\quad \frac{\text{Flow energy}}{m} = \frac{p \cdot Q}{m} = \frac{p}{\rho}\,(\text{joules/kg}) \tag{10.43}$$

$$2.\quad \frac{\text{Potential energy}}{m} = g \cdot z\ (\text{joules/kg}) \tag{10.44}$$

$$3.\quad \frac{\text{Kinetic energy}}{m} = \frac{1}{2}u^2\ (\text{joules/kg}) \tag{10.45}$$

10.3.2.5 Energy Head

When the energy terms are divided by the weight (m · g), energy per Newton will be the result. Studying the units closely:

$$\frac{J}{N} = \frac{N \cdot m}{N} = m\ (\text{metres})$$

This form of energy is normally referred to as the 'energy head' and the three forms of energy expressed in this way are as follows:

$$\frac{FE}{mg} = \frac{p}{\rho g} = h$$

$$\frac{PE}{mg} = z$$

$$\frac{KE}{mg} = \frac{u^2}{2g}$$

The term FE is referred to as the pressure head; previously, it was shown that (p/ρg) = h.

This is the height a liquid will rise to in a vertical pipe that is connected to the system. The potential energy term is the actual height relative to a datum level. The term $u^2/2g$ is called the kinetic head; this is the pressure head that results from converting the velocity into pressure.

10.3.3 Bernoulli's Equation

Bernoulli's equation is an expression of the conservation of energy. If no energy is added to the system such as work or heat, then the total energy of the fluid is conserved. Initially, the only forces considered are gravity, pressure and inertia forces. The viscosity forces are assumed to be negligible and the fluid is assumed to be a perfect inviscid fluid under steady flow conditions.

Referring to Figure 10.8, the total energy (E_t) at sections (1) and (2) must be equal to each other. So

$$E_t = p_1Q_1 + mgz_1 + m\frac{u_1^2}{2} = p_2Q_2 + mgz_2 + m\frac{u_2^2}{2} \tag{10.46}$$

Dividing by the mass gives the specific energy form:

$$\frac{E_t}{m} = \frac{p_1}{\rho_1} + gz_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + gz_2 + \frac{u_2^2}{2} \quad (10.47)$$

Dividing by 'g' will give the energy terms per unit weight:

$$\frac{E_t}{mg} = \frac{p_1}{g\rho_1} + z_1 + \frac{u_1^2}{2g} = \frac{p_2}{g\rho_2} + z_2 + \frac{u_2^2}{2g} \quad (10.48)$$

Now, since p/ρg = pressure head 'h', the total head is given by the following equation:

$$h_t = h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g} \quad (10.49)$$

This is Bernoulli's equation expressed in the form of 'h', in which each term is an energy head in metres. 'z' is the potential (or gravitational) head and $u^2/2g$ is the kinetic or velocity head.

When considering an actual system, there will be friction within the pipe and elsewhere. Heat is produced but will be absorbed by the fluid resulting in a rise in the internal energy and hence temperature. This temperature rise will be very small and if the pipe is long, the energy might be lost as heat transfer to the surroundings. As the equations do not include internal energy, the balance will be lost and an extra term is required on the right-hand side of the equation maintaining the balance. This term is either the head lost due to friction (h_L) or the pressure loss (p_L)

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g} + h_L \quad (10.50)$$

Expressing the equation in the form of pressure:

$$p_1 + \rho g z_1 + \frac{\rho u_1^2}{2g} = p_2 + \rho g z_2 + \frac{\rho u_2^2}{2g} + p_L \quad (10.51)$$

EXAMPLE 10.6

Figure 10.10 depicts a tank being filled via a pump delivering water through a pipe of 30 mm bore.

Determine the pressure at position '1' when the flow rate is 0.0014 m^3/s. The density of water is 1000 kg/m^3 and the pressure loss due to friction is 50 kPa.

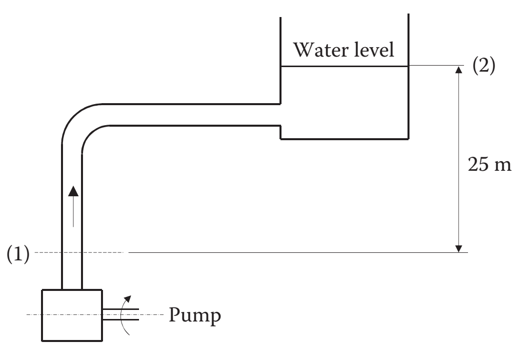


FIGURE 10.10 Example 10.6.

Solution

Internal area of the pipe:

$$A = \frac{\pi d^2}{4}$$
$$= \frac{\pi \times 0.03^2}{4}$$
$$= 706.86 \times 10^{-6}\ m^2$$

Flow rate (Q):

$$= 0.0014\ m^3/s$$

Mean velocity of water (u_m):

$$= \frac{Q}{A}\ m^2$$
$$= \frac{0.0014}{706.86 \times 10^{-6}}$$
$$= 1.981\,m/s$$

Applying Bernoulli's equation between position '1' and the water level in the tank:

$$p_1 + \rho g z_1 + \frac{\rho u_1^2}{2} = p_2 + \rho g z_2 + \frac{\rho u_2^2}{2} + p_L$$

Make position '1' the low level of the datum; therefore, 'z_1' = 0 and $z_2 = 25$ m.
The pressure at the water level will be zero-gauge pressure

$$p_L = 50{,}000\ Pa$$

The velocity of water at position '1' is 1.98 m/s and at the surface of the water level, the velocity is zero.

Therefore,

$$p_1 + 0 + \frac{1000 \times 1.98^2}{2} = 0 + 1000 \times 9.854 \times 25.0 + 0 + 50{,}000$$

Hence,

$$p_1 = 294.39\ kPa\ (Ans)$$

EXAMPLE 10.7

A horizontal nozzle is discharging water into an open vessel at atmospheric pressure. The inlet of the nozzle has a bore area of 600 mm^2 and the discharge has a bore area of 200 mm^2. Figure 10.11 shows the diagram of the nozzle. Calculate the water flow rate with the inlet pressure that is 400 Pa. Assume there are no energy losses.

Solution

Applying Bernoulli's equation between positions (1) and (2):

$$p_1 + \rho g z_1 + \frac{\rho u_1^2}{2} = p_2 + \rho g z_2 + \frac{\rho u_2^2}{2} + p_L$$

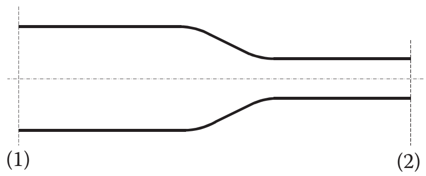


FIGURE 10.11 Horizontal venturi. (1) Inlet and (2) discharge.

Using the gauge pressure, $p_2 = 0$ and as the venture is horizontal, the potential terms cancel out. The loss term is also zero; hence, the equation simplifies to the following:

$$p_1 + \frac{\rho u_1^2}{2} = \frac{\rho u_2^3}{2}$$

From the continuity equation:

$$u_1 = \frac{Q}{A_1} = \frac{4Q}{\pi \times 0.6^2} = 3.537Q$$

$$u_2 = \frac{Q}{A_2} = \frac{4Q}{\pi \times 0.2^2} = 31.831Q$$

Substituting these values into Bernoulli's equation:

$$400 + 1000 \times \frac{(3.537Q)^2}{2} = 1000 \times \frac{(31.831Q)^2}{2}$$

$$400 + 6255.185Q^2 = 506606.3Q^2$$

Therefore,

$$400 = (506{,}606.3 - 6255.185)\ Q^2$$

$$\frac{400}{500{,}351.11} = Q^2$$

$$Q = 0.0283\ m^3/s\ \text{(Ans)}$$

10.3.4 Stagnation Point

Consider a circular bar located within a flowing fluid whether it is a liquid or gas. The fluid will hit the part and divide to go to either side of the body as shown in Figure 10.12.

At the point where the fluid parts, there will be an area in which the fluid will have zero velocity. Stagnation points exist at the surfaces of objects within the flow field where the fluid is brought to rest by the object. Wings on an aircraft are a further example, where the airstream parts at the leading edge to either flow above or below the wing surface, thereby providing lift.

In the flow stream, Bernoulli's equation shows that the energy comprising pressure + velocity + head is constant and as a result at the stagnation point, the pressure is increased from 'p' to $p + (\rho u^2/2)$ as the velocity energy is converted into pressure energy. For a uniform density fluid, the value of $p + (\rho u^2/2)$ is known as the stagnation pressure of the streamline.

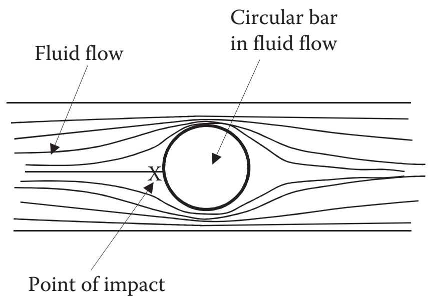


FIGURE 10.12 Circular bar in a fluid flow.

A manometer connected to the point 'X' (Figure 10.12) would indicate the stagnation pressure $((p/\rho g) + (u^2/2g))$. If the static head $(p/\rho g)$ is known, then by subtraction, the velocity head and hence the velocity can be easily calculated. This is the basis of operation for a Pitot tube to measure airspeed.

10.4 HYDROSTATICS

The definitions of hydrostatics include the following:

> The pressure exerted by a fluid at equilibrium at a given point within the fluid, due to the force of gravity. Hydrostatic pressure increases in proportion to the depth measured from the surface because of the increasing weight of the fluid exerting downward force from above.

In essence, the study of hydrostatics covers fluid bodies that are at rest or moving sufficiently slowly; so, there is no relative motion between the adjacent parts of the body.

There are only pressure forces acting perpendicular to any surface and there are no shear stresses.

10.4.1 Buoyancy

The principles of Archimedes state that the upthrust on a body fully or partly immersed in a fluid at rest is equal to the amount of fluid displaced.

A floating body will be in equilibrium under the action of the weight 'W' acting vertically downwards at its centre of gravity 'G' as shown in Figure 10.13. The upthrust 'R' acts vertically upwards through the centre of buoyancy 'B' for equilibrium.

Hence,

Weight of body 'W' = Upthrust 'R'
= Weight of liquid displaced

'R' and 'W' must act in the same vertical straight line.

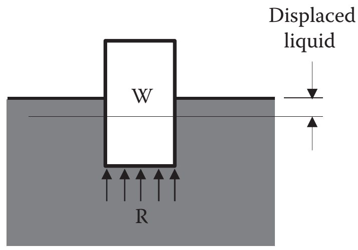


FIGURE 10.13 Archimedes' principle.

The surrounded fluid acting on an immersed body generates an upward thrust and this is known as the force of buoyancy. This thrust acts through the centroid of the displaced volume, and this is referred to as the centre of buoyancy. The centre of buoyancy is not the same as the centre of gravity that relates to the distributed weight within the body. If the body is solid with a uniform density identical to water and the body is fully immersed in water, the force of the buoyancy will be exactly equal to the weight, and in this case, the centre of buoyancy will be the same as the centre of gravity. The body will then be in equilibrium with the surrounding fluid.

This principle also applies to gasses as well as liquids. A balloon, filled with hydrogen having a lower density to that of the surrounding air at the sea level, will then rise to a height where the weight of air that is displaced equals the weight of gas in the balloon.

EXAMPLE 10.8

It is required to lay a steel pipeline for conveying gas across a stretch of water. The pipeline will be completely immersed in water and will be anchored at intervals of 3.0 m along its length. The details of the pipeline are

Internal diameter = 1.20 m
External diameter = 1.25 m

Calculate the buoyancy force in N/m run and the upward force acting on each anchorage. The density of steel is 7900 kg/m^3, and that of water is 1000 kg/m^3.

Solution

Buoyancy force/metre run = upthrust/metre run:

= Weight of water displaced/metre run

$$= (1000 \times 9.81) \times \frac{\pi \times 1.25^2}{4}$$

= 12.039 kN/m

The anchorages are 3.0 m apart; therefore

Upward force on anchorage = (buoyancy force – weight)

For 3.0 m length of pipe:

$$\text{The weight of 3 m length of pipe} = 3.0 \times (7900 \times 9.81) \times \frac{\pi(1.25 - 1.20)^2}{4}$$
$$= 22.369\ \text{kN}$$

Buoyancy force on 3.0 m length of pipe:

= 3.0 × 12.039

= 36.117 kN

Hence, upward force acting on the anchorage:

= (36.117 – 22.369) kN

= 13.748 kN (Ans)

In a situation where the centre of gravity (G) does not have the same location as the centre of buoyancy (B), the body will orientate itself so that the centre of gravity is below the centre of

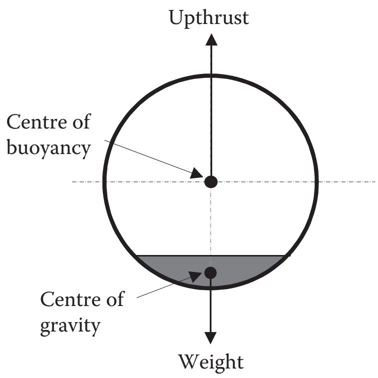


FIGURE 10.14 Centre of gravity.

buoyancy (see Figure 10.14). The figure considers a hollow vessel that has a heavy mass in a fixed position within the vessel. In this case, the centre of buoyancy is well above the centre of gravity for the vessel and as such, the vessel is in a stable equilibrium position.

If the vessel had a centre of buoyancy that was below the centre of gravity but on the same centreline as in Figure 10.15a, the vessel would, in theory, be in a stable position. If the vessel was displaced by a small angle, the vessel would then be in an unstable position; a couple would be generated to rotate the vessel to an equilibrium position where the centre of buoyancy will be above the centre of gravity as in Figure 10.15b.

The definition for stable and unstable equilibrium can be simply stated as when an immersed body, initially at rest, is displaced so that the force of buoyancy and the centre of gravity are not on the same vertical line:

1. The body is stable if the resulting couple tends to bring the body back to its original position.
2. The body is unstable if the resulting couple moves the body away from its original position.

10.4.2 Metacentre and Metacentre Height

In Figure 10.16a, a rectangular vessel is immersed in water where the centre of buoyancy is at position 'B', the centre of gravity is at 'G' and the waterline is at 'S:S'. Consider the vessel that is now tilted to the position shown in Figure 10.16b where the waterline is now at 'S:S'. The centre of buoyancy has now moved to position 'B_1'. There is an upthrust 'W' due to the buoyancy at 'B_1' and

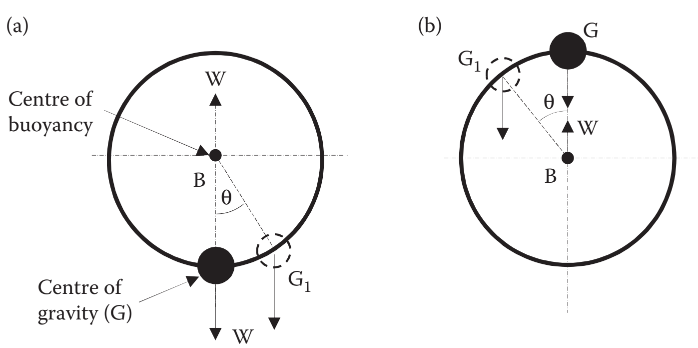


FIGURE 10.15 Centre of gravity. (a) Stable and (b) unstable.

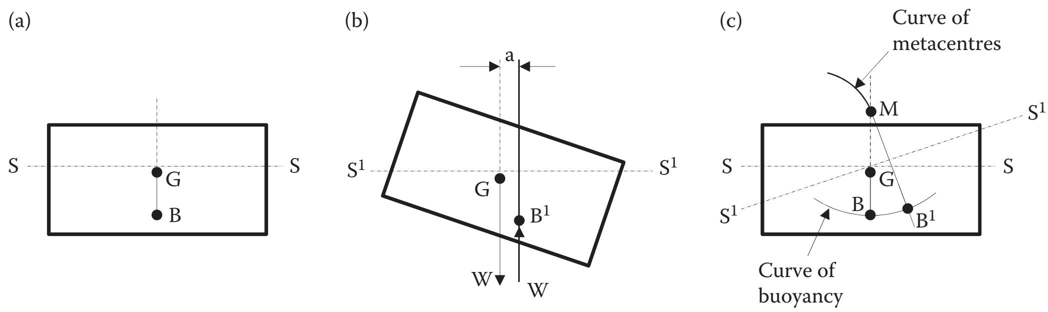


FIGURE 10.16 Metacentre height.

the weight of the vessel 'W' is acting down at 'G' (the centre of gravity). A couple 'W · a' is acting to restore the vessel back to its original equilibrium position.

Drawing a line vertically to intersect the line joining the centre of buoyancy and the centre of gravity will give the position of the metacentre for the position of the vessel shown. Heeling the vessel further will produce different metacentre positions. Provided that the metacentre remains above the centre of gravity, the vessel will remain stable and will return to its equilibrium position.

The metacentre height is the distance between the metacentre and the centre of gravity.

If the vessel is tilted far enough where the metacentre is below the centre of gravity, then the vessel will become unstable and could result in capsizing.

From the above discussion, the following conclusions can be made:

- A floating vessel is stable provided that the metacentre lies above the centre of gravity 'G'.
- A floating vessel is in neutral equilibrium if the metacentre lies on the centre of gravity 'G'.
- A floating vessel is unstable if the metacentre lies below the centre of gravity 'G'.

EXAMPLE 10.9

A vessel has a displacement of 2.5×10^6 kg in freshwater. A mass of 20×10^3 kg is moved across the deck of the vessel causing a tilt of 4.4°.

Calculate the transverse metacentre height.

SOLUTION

The overturning moment created when a mass of 20×10^3 kg is moved 9 m across the deck

$$= (20 \times 10^3 \times 9.81) \times 9 \text{ N/m}$$

$$= 1.7658 \times 10^6 \text{ N/m}$$

The restoring moment:

$= W \cdot x$
$= W \cdot GM \cdot \theta$ (GM is the distance between the centre of gravity and the metacentre.)

Since $W = 2.5 \times 10^6 \times 9.81$ N and $\theta = 4.4°$.

The restoring moment

$$= 2.5 \times 106 \times 9.81 \times 4.4° \times GM$$

The vessel is in equilibrium while in the tilted position.

The overturning moment represents the righting moment:

$$20 \times 10^3 \times 9.81 \times 9 = 2.5 \times 10^6 \times 4.4° \times GM$$

Metacentric height (GM):

$$= \frac{20 \times 9}{2500 \times 0.07667}$$
$$= 0.939 \text{ m (Ans)}$$

10.4.3 Pressure in Liquids

A perfect fluid cannot oppose or apply any shear force and is defined as non-viscous (or inviscid) under all conditions. The intensity of normal forces is called the pressure and will be positive if the fluid is compressed.

Consider a small element of fluid of a uniform width that is subject to pressures p_x, p_y and p_z as shown in Figure 10.17. It is assumed that the element is so small that the pressures are assumed to be uniform across the faces of the element (gravity is not considered).

Equating forces:

$$P_z \cdot A \cdot \sin\theta = A\sin\theta P_y \quad (10.52)$$

$$P_z \cdot A \cdot \cos\theta = A\cos\theta P_x \quad (10.53)$$

This example illustrates a perfect liquid and to a large extent, for real fluids, the pressure at a point will be the same in all directions (or the element would move in the direction of least pressure). If no other forces are acting on the body of the fluid, the pressure must be the same at all neighbouring points. Therefore, in this case, the pressure will be the same throughout the fluid and will be the same in any direction at a point (Pascal's principle). In SI units, pressure is expressed in N/m^2 (Pascal).

A practical application of the above principle is seen in the humble car jack where a small piston can exert sufficient force to raise an automobile off the road. Figure 10.18 illustrates the principle behind this.

In the figure, it is seen that a small piston and cylinder (A) are connected to a much larger piston and cylinder (B). When a force is applied to the piston (A), the pressure in the fluid (A) will be

$$P_A = \frac{F_A}{A_A} \quad (10.54)$$

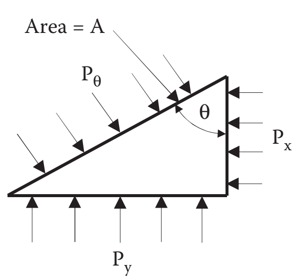


FIGURE 10.17 Pressure in liquids.

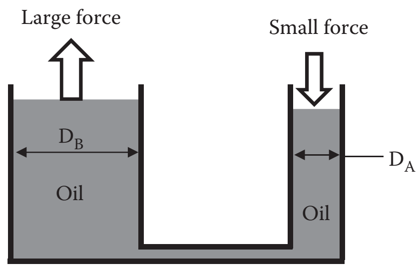


FIGURE 10.18 Force amplification.

where
P_A is the pressure in cylinder (A)
F_A is the force from piston (A)
A_A is the area of cylinder (A)

As the two cylinders are interlinked, the fluid pressure in cylinder (B) will be equal to that of cylinder (A) according to Pascal's law.

That is

$$P_A = P_B \tag{10.55}$$

Therefore, the force generated by piston (B) will be

$$F_B = A_B \times P_A \tag{10.56}$$

or

$$F_B = \frac{A_B}{A_A} F_A \tag{10.57}$$

Distance moved:

The distance 'h' moved by piston (A) will displace a volume of fluid:

$$\Delta V = h_A \cdot A_A \tag{10.58}$$

The piston (B) will move a distance:

$$h_B = \frac{A_A}{A_B} h_A \tag{10.59}$$

EXAMPLE 10.10

It is required to lift a part of a vehicle to facilitate changing a wheel. The vehicle weight is 1.5 t.

Design a hydraulic jack that will lift a mass of 560 kg using a handle of 300 mm long with a hand force of approximately 40 N (4.07 kg).

Figure 10.19 depicts the arrangement of the jack that is fitted with a non-return valve to enable a suitable height to be obtained with steady pumping.

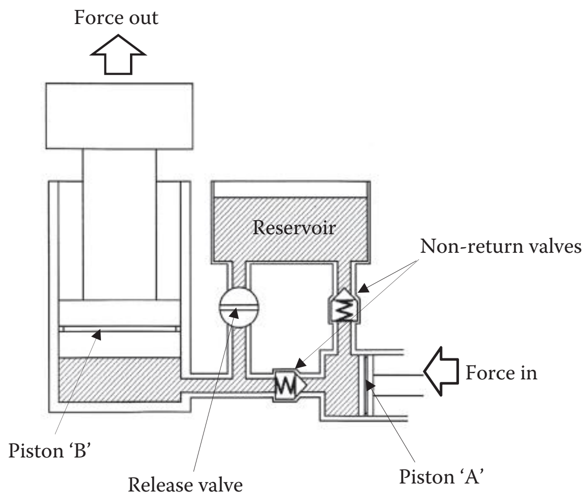


FIGURE 10.19 Example 10.10.

Solution

Vehicle weight = 1.5 t.

Assuming the jack carries a quarter of this weight and allowing a safety factor of 2 for the proof load:

$$\text{Jack proof load} = \frac{1500\ \text{kg}}{4} \times 2$$
$$= 750\ \text{kg}$$

The force to be exerted by the lifting piston:

$$= 751 \times 9.81\ \text{N}$$
$$= 7354.98\ \text{N}$$

The diameter of the lifting cylinder:

$$= 50.0 \times 10^{-3}\ \text{m}$$

$$\text{Area } (A_L) = \frac{\pi \times (50.0 \times 10^{-3})^2}{4}$$
$$= 1.9635 \times 10^{-3}\ \text{m}^2$$

The pressure in the lifting cylinder (P_L):

$$P_L = \frac{\text{lifting force}}{\text{area of lifting cylinder}}$$
$$= \frac{7354.98\ \text{N}}{1.9635 \times 10^{-3}\ \text{m}^2}$$
$$P_L = 3.746 \times 10^6\ \text{Pa}$$

The diameter of the pumping cylinder (Dia'_p)

$$Dia'_p = 12.0 \times 10^{-3}\ \text{m}$$

The area of the pumping cylinder (A_p):

$$A_p = \frac{\pi \times (12.0 \times 10^{-3})^2}{4}$$
$$= 113.097 \times 10^{-6}\ m^2$$

The force required on the pumping cylinder (F_p):

$$F_p = \text{pressure} \times A_p$$
$$= 3.746 \times 10^6\ Pa \times 113.097 \times 10^{-6}\ m^2$$
$$= 423.647\ N$$

Pumping force F_A: Taking moments about a pivot point offset 25.0 mm from the centre of piston 'A' (Figure 10.19):

$$F_A = \frac{25.0 \times 10^{-3}\ mm \times 423.647\ N}{300 \times 10^{-3}\ mm}$$
$$F_A = 35.3\ N\ (3.6\ kg)$$

The stroke of the pumping cylinder (h_p) will be approximately 12 mm and the stroke of the lifting cylinder (h_L) will be

$$h_L = \frac{area_p}{area_L} \times h_p$$
$$= \frac{113.097 \times 10^{-6}}{1.9635 \times 10^{-3}} \times 12.0 \times 10^{-3}$$
$$= 0.691 \times 10^{-3}\ m \quad \text{say } 0.690\ mm$$

10.4.4 Pressure Due to the Weight of a Liquid

If a tank with a cross-sectional area 'A' and a depth 'h' is open to the atmosphere, the surface of the liquid will be at atmospheric pressure. If pressure measurements are made at various depths, the pressure will be found to increase linearly as shown in Figure 10.20.

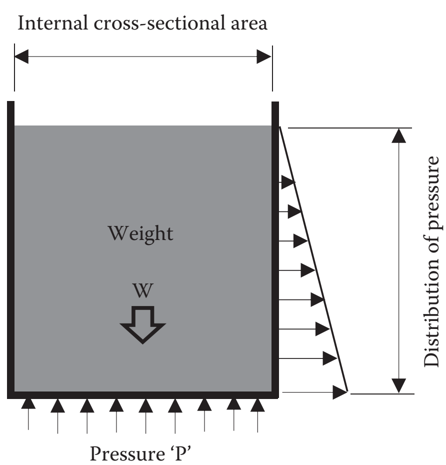


FIGURE 10.20 Pressure due to the weight of a fluid.

The volume of the liquid is

$$V = Ah \text{ m}^3 \tag{10.60}$$

The mass of the liquid will be

$$\begin{aligned} m &= \rho V \\ &= \rho Ah \text{ kg} \end{aligned}$$

Multiplying the mass by the gravitational constant 'g':

$$\begin{aligned} W &= mg \\ &= \rho Ahg \text{ Newton} \end{aligned}$$

It follows that the pressure at the bottom of the tank will be

$$p = \frac{W}{A} \text{ N/m}^2$$

The pressure at a depth 'h' in a liquid is given by the following equation:

$$p = \rho gh \text{ N/m}^2 \text{ (pascal)} \tag{10.61}$$

EXAMPLE 10.11

A water tank is fitted with a circular inspection hatch of 0.75 m diameter and is located on the bottom of the tank at a depth of 5 m. The density of water is 1000 kg/m^3.

Calculate the force acting on the hatch due to the water pressure.

SOLUTION

The pressure at the bottom of the tank:

$$p = \rho gh$$

where

$\rho = 1000$ kg/m^3
$g = 9.81$ m/s^2
$h = 5$ m

$$\begin{aligned} p &= 1000 \times 9.81 \times 5 \\ &= 49.05 \text{ kPa} \end{aligned}$$

The force acting on the hatch is a product of the hatch area and pressure.

$$\begin{aligned} A &= \frac{\pi d^2}{4} \\ &= \frac{\pi \times 0.75^2}{4} \\ &= 0.4418 \text{ m}^2 \end{aligned}$$

The force 'F' = p · A:

$$= 49.05 \times 10^3 \times 0.4418$$
$$= 21.67 \times 10^3 \text{ N}$$
$$= 21.67 \text{ kN}$$

10.4.5 Forces on Submerged Surfaces

A section submerged vertically below the surface of a liquid will have variable force acting on it due to the varying pressure of the liquid with the increase in depth.

Figure 10.21 shows a vertical rectangular section fully immersed in a liquid.

From Section 10.4.4, the pressure at a depth 'h' in a liquid is given by Equation 10.61:

$$p = \rho gh$$

In this example, 'h' is being replaced by 'y' to denote depth for reasons that will become obvious. Hence,

$$p = \rho gy \tag{10.62}$$

The force acting on the elemental strip due to this pressure is

$$dF = p\ dA = \rho Bgy\ dy \tag{10.63}$$

The total force acting on the surface due to the pressure is denoted by 'F' and is obtained by integrating this expression between the limits y_1 and y_2.

$$F = \rho gB\left(\frac{y_2^2 - y_1^2}{2}\right)$$

This expression can then be factored:

$$F = \rho gB\left[\frac{(y_2 - y_1)(y_2 + y_1)}{2}\right]$$

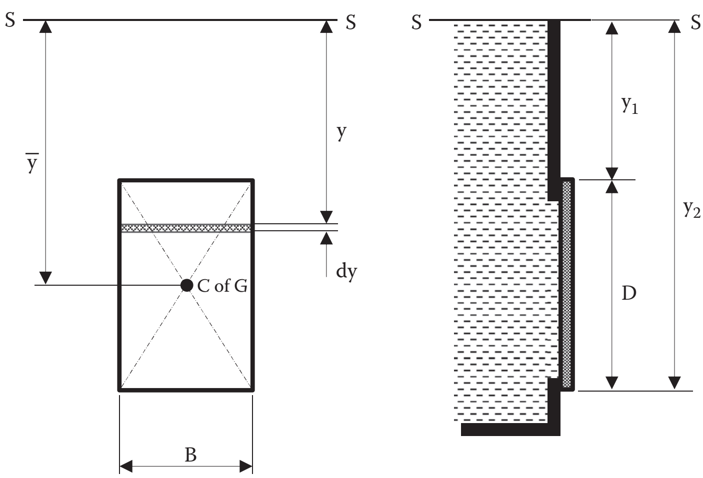


FIGURE 10.21 Submerged section.

Now, $(y_2 - y_1) = D$ (the depth of the surface); so, $B(y_2 - y_1) = B \cdot D$ = total area of the surface. $(y_2 - y_1)/2$ is the distance from the free surface to the centroid $\bar{y}$.

It follows that the total force is given by the expression:

$$F = \rho g A \bar{y} \tag{10.64}$$

The term 'Ay' is the first moment of area and the total force on the submerged surface is

$$F = \rho g \times \text{first moment of area about the free surface}$$

Note: The first moment of area is the area x the distance from the centroid of the area to an axis, in this case, the free surface 'S:S'.

10.4.6 Centre of Pressure

The centre of pressure is that point on which it is considered that the total force is assumed to act on a submerged section. Referring to Figure 10.21, the force acting on the strip is 'dF'. This force produces a turning moment on the section with respect to the axis 'S:S'. The turning moment due to dF is

$$dM = y\,dF = \rho g B y^2\,dy \tag{10.65}$$

The total turning moment about the axis 'S:S' due to the pressure is found by integrating Equation 10.65 between the limits y_1 and y_2.

Hence

$$M = \int_{y_1}^{y_2} \rho g B y^2\,dy$$

$$= \rho g B \int_{y_1}^{y_2} y^2\,dy$$

By definition:

$$I_{ss} = B \int_{y_1}^{y_2} y^2\,dy$$

Therefore

$$M = \rho g I_{ss}$$

This moment should also equal to that obtained from the total force 'F' multiplied by the distance '$\bar{h}$'. The position at the depth 'h' is called the 'centre of pressure'.

$\bar{h}$ is determined by equating the moments:

$$M = \bar{h} \cdot F$$

$$= \bar{h} \rho g A \bar{y}$$

$$= \rho g I_{SS}$$

$$\bar{h} = \frac{\rho g I_{SS}}{\rho g A\bar{y}} = \frac{I_{SS}}{A\bar{y}}$$

$$\bar{h} = \frac{\text{second moment of area}}{\text{first moment of area}} \text{ about 'S:S'} \quad (10.66)$$

To resolve this expression requires a degree of experience with the parallel axis theorem that is required to resolve the second moment of area about the free surface.

The parallel axis theorem is as follows:

$$I_{SS} = I_{CG} + A\bar{y}^2$$

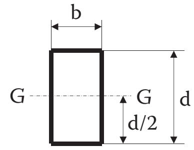
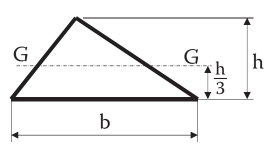
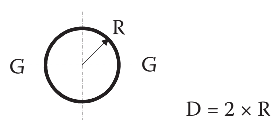
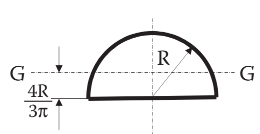
where

I_{SS} is the second moment about the free surface S:S

I_{CG} is the second moment of area about the centroid C of G

Table 10.3 gives some geometric properties of common figures.

TABLE 10.3
Geometric Properties for Some Common Figures

		Area 'A'	2nd Moment of Area I_{gg} about Axis G:G through the Centroid
Rectangle		bd	$\frac{bd^3}{12}$
Triangle		$\frac{b \cdot h}{2}$	$\frac{b \cdot h^3}{36}$
Circle		$\frac{\pi D^2}{4}$	$\frac{\pi D^4}{64}$
Semi-circle		$\frac{\pi R^2}{2}$	$0.1102\ R^4$

EXAMPLE 10.12

A rectangular inspection plate is fitted onto a steel water-storage tank as shown in Figure 10.22.

Calculate the total force acting on the inspection plate and the position of the centre of pressure. Also, calculate the total moment about the bottom edge of the plate. Assume the water density is 1000 kg/m³.

Solution

Total force $F = \rho g A\bar{y}$

For the plate:

$$\bar{y} = 0.75 + 0.5\,(\text{m})$$
$$= 1.25\,\text{m}$$
$$\text{Area} = 1.0 \times 0.75\,(\text{m}^2)$$
$$= 0.75\,\text{m}^2$$
$$F = 1000 \times 9.81 \times 0.75 \times 1.25$$
$$= 9.197\,\text{kN}$$

$$\bar{h} = \frac{\text{second moment of area}}{\text{first moment of area}}$$

First moment of area:

$$= A\bar{y} = 0.75 \times 2 = 1.5\,\text{m}^3$$

Second moment of area:

$$I_{SS} = \left(\frac{BD^3}{12}\right) + Ay^2$$
$$= \left(\frac{0.75 \times 1.0^3}{12}\right) + (0.75 \times 1.25^2)$$
$$= 1.234\,\text{m}^4$$

$$\bar{h} = \frac{I_{SS}}{A\bar{y}}$$
$$= 0.823\text{m}$$

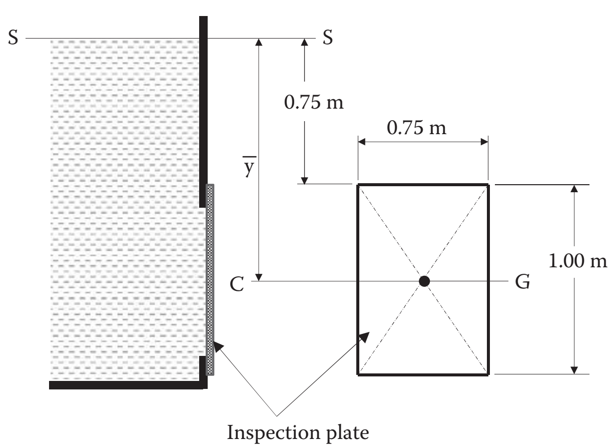


FIGURE 10.22 Example 10.12.

The distance from the bottom edge:

$$x = 2.5 - 0.823$$
$$= 1.677 \text{ m}$$

The moment about the bottom edge:

$$= Fx$$
$$= 9.197 \times 10^3 \times 1.677$$
$$= 15.42 \text{ kN m}$$

10.5 DIMENSION ANALYSIS

The United Kingdom has adopted the system of metric units known as the Système International d'Units, abbreviated to SI. In some old designs imperial units are still being used, these are being phased out as these designs are updated.

The SI has six basic units; these are arbitrarily defined as

Length	Metre (m)
Mass	Kilogram (kg)
Time	Second (s)
Temperature	Degree (θ)
Electric current	Ampere (A)
Luminous intensity	Candela (cd)

The last two items are not used in fluid mechanics, temperature is only used infrequently.

All other units are derivations of these four fundamental units, M, L, T and θ. Other systems of units could be used but staying with M, L, T, θ removes any constraints to a particular system of measurement.

10.5.1 Dimensions

In mechanics, all quantities can be expressed in terms of these fundamental dimensions of mass M, length L and time T.

Hence

$$\text{Acceleration} = \frac{\text{Distance}}{(\text{Time})^2}$$

So that

$$\text{Dimensions of acceleration} = \frac{\text{Dimension of distance}}{(\text{Dimension of time})^2} = \frac{L}{T^2}$$

So that

$$\text{Dimension of force} = \text{Dimension of mass} \times \text{dimension of acceleration}$$
$$= \frac{ML}{T^2}$$

TABLE 10.4
Dimensions of Some Common Quantities

Quantity	Description	Dimension
Length, size	Include all linear measurements	L
Area	Length × length	L^2
Volume	Area × length	L^3
First moment of area	Area × length	L^3
Second moment of area	Area × length2	L^4
Angle	A ratio; arc/radius	1
Strain	A ratio	1
Time		T
Velocity	Distance/time	LT^{-1}
Angular velocity	Angle/time	T^{-1}
Acceleration	Velocity/time	LT^{-2}
Angular acceleration	Angular velocity/time	T^{-2}
Kinematic viscosity	Dynamic viscosity/mass density	L^2T^{-1}
Mass		M
Force	Mass × acceleration	MLT^{-2}
Weight	Force	MLT^{-2}
Mass density	Mass/volume	ML^{-3}
Specific weight	Weight/volume	$ML^{-2}T^{-2}$
Pressure (intensity)	Force/area	$ML^{-1}T^{-2}$
Shear stress	Force/area	$ML^{-1}T^{-2}$
Elastic modulus	Stress/strain	$ML^{-1}T^{-2}$
Impulse	Force/time	MLT^{-1}
Momentum	Mass × velocity	MLT^{-1}
Work, energy	Force × distance	ML^2T^{-2}
Power	Work/time	ML^2T^{-3}
Moment of a force	Force × distance	ML^2T^{-2}
Dynamic viscosity	Shear stress/velocity gradient	$ML^{-1}T^{-1}$

And when written down in dimensional form:

$$= MLT^{-2}$$

Table 10.4 shows the dimensions of common quantities.

10.5.2 Dimensional Equations

When an equation is to represent a physical or real quantity, the terms on both sides must be of the same sort. For example, all forces as well as both sides should be numerically equal, otherwise, the equation will be meaningless. Every term must have the same dimensions, so that like is compared with like.

As an example, consider the equation $v^2 = u^2 + 2as$.

This equation calculates the final velocity 'v' of a mass which starts with an initial velocity 'u' and then receives an acceleration of 'a' for a distance of 's'. When dimensions are substituted for the quantities, each term must have the same dimensions if the equation is to be true.

The dimensions of the quantities are

$$v = LT^{-1}$$
$$u = LT^{-1}$$
$$a = LT^{-2}$$
$$s = L$$

The dimensions of v^2 are $(LT^{-1})^2 = L^2T^{-2}$.
The dimensions of u^2 are $(LT^{-1})^2 = L^2T^{-2}$.
The dimensions of 2 as are $(LT^{-2}) \times L = L^2T^{-2}$.

All three dimensions have the same form and therefore the equation is dimensionally correct and therefore could represent a real event.

There are other quantities that require more careful consideration when writing in the basic MLTθ form. Force is one such important unit. In SI, the unit of force is the 'N'. Engineers have agreed to define force as the need to accelerate 1 kg mass at an acceleration of 1 m/s^2. The Newton is a derived unit that is equal to 1 kg m/s^2. In a dimensional form, the dimensions of force becomes MLT^{-2} and this has to be considered when writing dimensions containing force.

EXAMPLE 10.13

Deduce the basic dimensions for pressure.

Solution

The definition of pressure is

$$p = \frac{\text{force}}{\text{area}}$$

Pascal is the SI unit for pressure and the units are N/m^2.
The dimensions for force is MLT^{-2} and for area is L^2. Therefore, the basic dimensions for pressure is

$$p = \frac{MLT^{-2}}{M^2}$$

This can be written in standard form:

$$p = ML^{-1}T^{-2} \text{ (Ans)}$$

10.6 FLUID DRAG

If an object is placed in a flowing fluid, a resistance to the flow will be generated causing the object to be pulled by the fluid in the direction of the fluid flow. This resistance is known as 'drag' and is a function of the form of the object and skin friction between the object and the fluid.

1. Form drag (or pressure drag) is based on the pressure difference between the upstream and downstream surfaces of the object.
2. Skin friction resulting from the viscous shear of the fluid flowing over the object's surfaces.

10.6.1 Form Drag

Form or pressure drag is applicable to bodies that are tall in comparison with their length in the direction of the flow. Figure 10.23a depicts such a body in a flowing fluid and these bodies are

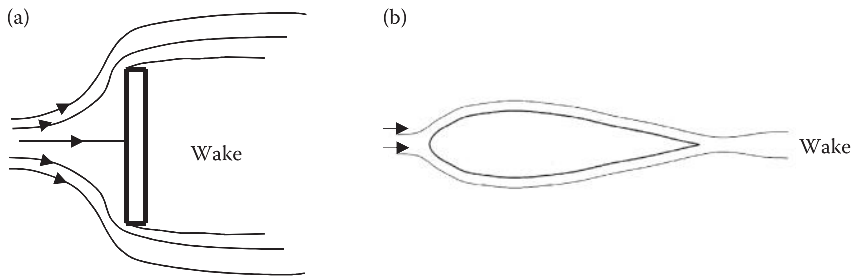


FIGURE 10.23 (a) Bluff and (b) streamlined bodies.

known as 'bluff bodies'. Figure 10.23b shows a streamlined body in a flowing fluid and as is seen there is very little disturbance to the flow around the body.

Examples of bluff bodies would be the valve plate in a shut-off valve and the leg of a pier or bridge support in a river. In this instance, the fluid locally speeds up around the leading edge of the body and the boundary layer quickly breaks away from the surface. The fluid is sucked in from behind the body in the opposite direction. This creates a wake behind the body and this in turn creates a lower pressure region resulting in a significant form drag.

If the body was shaped as in Figure 10.23b, the amount of drag produced by the section would be appreciably reduced, the low-pressure region will be reduced, and the wake produced will also be smaller. If a road vehicle is fitted with a streamlined body, it is able to attain a higher speed than without it, showing the significant reduction in form drag.

10.6.2 Skin Friction Drag

If a flat plate is immersed in a flowing fluid with the plate orientated to be parallel to the flow, when the flow hits the leading edge of the plate the flow is retarded and a thin layer is formed by the fluid flowing along one side of the flat plate as shown in Figure 10.24a and b. The flow close to the surface of the plate slows down and the flow in this region is considered laminar. This slowing of the fluid is due to the viscous nature of the fluid.

The layers above the surface are moving therefore shearing is taking place between the layers of the fluid. The shear stress acting between the surface and the first moving layer is called 'the wall shear stress' and is denoted by the symbol 'τ_w'.

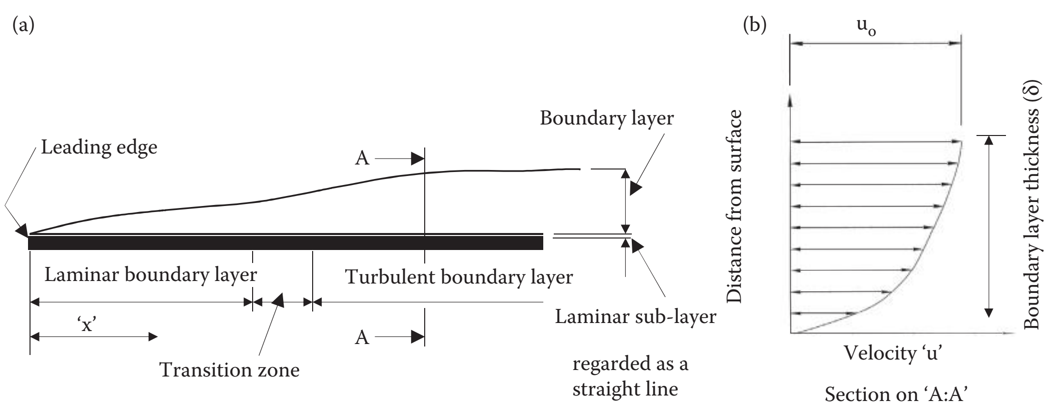


FIGURE 10.24 (a) and (b): Boundary layer.

The velocity of the fluid is increasing from zero at the surface of the plate to a maximum 'u_o' at a distance 'δ' above it. This layer is called the 'Boundary Layer' and 'δ' is the boundary layer thickness. Figure 10.24b shows how the velocity 'u' varies with the thickness of the layer 'y'.

Refer to Figure 10.24a again. At the leading edge of the plate, as the flow is retarded due to the viscous resistance and the boundary layer is originated, more flow is slowed down due to the viscous forces and the boundary layer then begins to thicken. At some point, the laminar flow loses stability and the flow becomes less even. This point where the laminar flow deteriorates is known as the 'transition point' and is at the start of the region called the transition region where the laminar flow changes to turbulent flow. At the plate surface, a thin laminar sub-layer will remain below the turbulent boundary layer.

The boundary layer thickness 'δ' grows in thickness with distance from the leading edge. At some distance from the leading edge, the layer reaches a constant thickness and then it is called a 'fully developed boundary layer'. The Reynolds number in these cases is

$$(R_e)_x = \frac{\rho u_o x}{\mu} \tag{10.67}$$

where x is the distance from the leading edge.

At low Reynolds numbers, the boundary layer may be laminar throughout the entire thickness of the layer. At higher Reynolds numbers, the flow is turbulent meaning at some point from the leading edge the flow within the boundary layer becomes turbulent.

A turbulent boundary layer is very unstable where the streamlines do not remain parallel to each other. The shape of the boundary layer represents an average of the velocity of the fluid at any height.

Between the laminar and turbulent section, there is a transition. The turbulent boundary layer exists on top of a thin laminar known as the 'laminar sub-layer'. The velocity gradient within this layer is linear. Further analysis would show that, for long surfaces, the boundary layer will be turbulent over most of its length. A number of equations have been developed to describe the shape of the laminar and turbulent boundary layers.

10.6.3 Estimating Skin Drag

Skin drag is a result of the wall shear stress 'τ_w' and acts on the wetted surface area. The drag force is

$$R = \tau_w \times \text{wetted area} \tag{10.68}$$

The dynamic pressure is the pressure that results from the conversion of the kinetic energy of the fluid into pressure energy and is defined by the expression $\rho u_o^2/2$. The drag coefficient is defined as

$$\begin{aligned} C_{Df} &= \frac{\text{Drag force}}{\text{Dynamic pressure} \times \text{wetted area}} \\ &= \frac{2R}{\rho u_o^2 \times \text{wetted area}} \\ &= \frac{2\tau_w}{\rho u_o^2} \end{aligned} \tag{10.69}$$

EXAMPLE 10.14

Calculate the drag force on both sides of a thin smooth rectangular plate 1.0 m wide × 2.0 m long where the 2.0 m length is parallel to the flow of fluid having a velocity of 40 m/s. The density of the fluid is 1000 kg/m^3 and its dynamic viscosity is 8 cP.

Solution

$$(R_e)_x = \frac{\rho u_o L}{\mu}$$
$$= \frac{1000 \times 40 \times 2}{0.008}$$
$$= 10 \times 10^6$$

$$C_{Df} = 0.074 \times (10 \times 10^6)^{-1/5}$$
$$= 0.00295$$

$$\text{Dynamic pressure} = \frac{\rho u_o^2}{2}$$
$$= \frac{1000 \times 40^2}{2}$$
$$= 800\ \text{kPa}$$

$$t_w = \text{CDf} \times \text{dynamic pressure}$$
$$= 0.00295 \times 800 \times 10^3$$
$$= 2360\ \text{Pa}$$
$$R = t_w \times \text{wetted area}$$
$$= 2360 \times 2 \times 1.0$$
$$= 4720.0\ \text{N (Ans)}$$

10.6.4 General Notes on Drag Coefficients

In fluid dynamics, the drag coefficient is commonly denoted by c_d and is a dimensionless quantity. This coefficient is used to quantify the resistance of an object in a flowing fluid and is a helpful guide in understanding the importance of the shape of an object in drag resistance. It is also useful for comparing various shapes.

Table 10.5 shows the drag coefficients for some common shapes.

A general approach to solving form drag which is due to pressure changes only and the drag coefficient due to the pressure only is represented by C_{dp} and is defined as

$$C_{dp} = \frac{\text{Drag force}}{\text{Dynamic pressure} \times \text{projected area}} = \frac{2R}{\rho u_o^2 \times \text{projected area}} \tag{10.70}$$

The projected area is the outline of the shape that is directly facing the fluid flow and the pressure that is acting on any point of the surface of the shape is 'p'. The force that is being exerted by the pressure on a small surface area will be p dA. If the shape is inclined at an angle 'θ' with the fluid flow, the force will be

$$p = \cos\theta\ dA \tag{10.71}$$

The total force can be found by integrating the projected surface.

$$R = \oint p \cos\theta\ dA \tag{10.72}$$

TABLE 10.5
Measured Drag Coefficients for Some Common Shapes

Shape	Figure	Drag Coefficient
Sphere		0.47
Half-sphere		0.42
Cone		0.50
Cube		1.05
Angled cube		0.80
Long cylinder		0.82
Short cylinder		1.15
Streamlined body		0.04
Streamlined half-body		0.09

The pressure distribution over the surface is expressed in the form of a pressure coefficient defined as follows:

$$C_p = \frac{2(p - p_o)}{\rho u_o^2} \qquad (10.73)$$

where
p_o is the static pressure of the undisturbed flowing fluid
u_o is the velocity of the undisturbed flowing fluid
$\rho u_o^2/2$ is the dynamic pressure of the stream

If any streamline that is affected by the surface is considered, by applying Bernoulli's theorem between an undisturbed point and any point on the surface, the following will result:

$$p_o + \frac{\rho u_o^2}{2} = p + \frac{\rho u^2}{2}$$

$$p - p_o = \frac{r}{2}(u_o^2 - u^2) \qquad (10.74)$$

$$C_p = \frac{2(p - p_o)}{\rho u_o^2}$$

$$= \frac{2\left(\frac{p}{2}(u_o^2 - u^2)\right)}{\rho u_o^2}$$

$$= \frac{(u_o^2 - u^2)}{u_o^2}$$

$$= 1 - \frac{u^2}{u_o^2} \tag{10.75}$$

The equation shows that when $u < u_o$, the pressure will be positive and when $u > u_o$, the pressure will be negative.

10.6.5 Total Drag

As explained previously, drag is made up of two components:

1. Skin friction drag
2. Form drag

This applies to all vehicles including aircraft and automobiles as well as bluff objects such as squares, cylinders and spheres. Calculating the drag forces on an object using theoretical methods is difficult. The majority of the data is captured using experimental methods including wind tunnel analysis and the concept of a drag coefficient is widely used.

The 'drag coefficient' is denoted by C and is defined by

$$C_d = \frac{\text{Resistance force}}{\text{Dynamic pressure} \times \text{projected area}}$$

$$C_d = \frac{2R}{\rho u_o^2 \times \text{projected area}} \tag{10.76}$$

EXAMPLE 10.15

A cylinder 100 mm diameter × 350 mm long is in a fluid stream moving at 0.75 m/s. The axis of the cylinder is normal to the direction of the fluid flow. The density of the fluid is 1000 kg/m^3. The drag force is found to be 75 N. At a point on the surface of the body, the pressure is measured at 90 Pa above the ambient pressure.

Calculate

1. The drag coefficient.
2. The velocity at the specified point on the surface.

Solution

$$\text{Projected area} = 0.10 \times 0.35 \text{ m}^2 = 0.035 \text{ m}^2$$

$R = 75$ N
$u_o = 0.75$ m/s
$\rho = 1000$ kg/m^3

Dynamic pressure (P_d):

$$P_d = \frac{\rho \cdot u^2}{2}$$
$$= \frac{1000 \times 0.75^2}{2}$$
$$P_d = 281.25\ \text{Pa}$$

$$C_d = \frac{\text{Resistance force}}{P_d \times \text{projected area}}$$
$$= \frac{75}{281.25 \times 0.035}$$
$$C_d = 7.619$$

$$p - p_o = \frac{\rho}{2}(u_o^2 - u^2)$$
$$90 = \frac{1000}{2}(0.75^2 - u^2)$$

Rearranging the equation to solve for 'u':

$$u = \sqrt{u_o^2 - \left(\frac{2(p - p_o)}{r}\right)}$$
$$= 0.618\ \text{m/s}$$

10.6.6 Drag on a Cylinder

From Equation 10.76, the drag coefficient is defined as

$$C_d = \frac{2R}{\rho u_o^2 \times \text{projected area}}$$

where the projected area is the length × diameter of the cylinder.

The flow pattern in the wake of the cylinder is dependent upon the Reynolds number and is illustrated in Figure 10.25a–d. The figure considers an infinitely long cylinder in a fluid flow that provides a two-dimensional flow pattern.

1. At very low velocities, where the Reynolds number is $R_e < 0.5$, the inertial forces are very low compared with the viscous forces and the streamlines return to the flow pattern behind the cylinder and the drag is roughly proportional to u_n. The drag is inversely proportional to C_d.
2. Between the Reynolds numbers 2–30, a wake begins to form behind the cylinder. The stream lines come together behind the cylinder but small eddies are formed which rotate in opposite directions.
3. With the Reynolds numbers between 40 and 70, the eddies elongate and the wake flow becomes unstable. As the Reynolds number increases to 70, the eddies elongate and begin to break away alternatively from each side of the cylinder.
4. At the higher levels of the Reynolds numbers ($R_e > 90$), the eddies begin to form vortices downstream from the cylinder. It is this unsymmetrical behaviour of the flow pattern that gives rise to vortex shedding and is also known as the Karman vortex street.

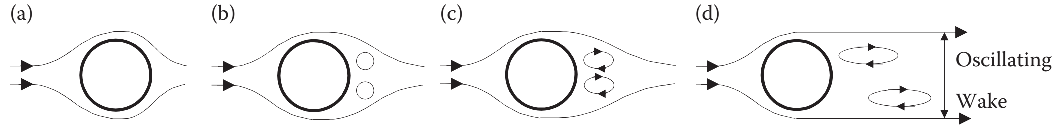


FIGURE 10.25 Fluid flow around a cylinder. (a) $R_e < 0.5$, (b) $R_e = 2$ to 30, (c) $R_e = 40$ to 70 and (d) $R_e > 90$.

The drag coefficient (C_d) reaches a minimum of approximately 0.9 at $R_e = 2000$. It then increases slightly due to the increasing turbulence of the wake and this position of separation gradually moves upstream. The profile drag at this stage is nearly all due to the pressure (form) drag.

As the Reynolds number reaches about 20×10^5, the laminar boundary layer becomes more turbulent before separation. The turbulent boundary layer has a higher kinetic energy than the laminar-based layer and as such is better able to withstand the adverse pressure gradient.

There is a sudden drop in the drag dropping to about a C_d of 0.3.

With a further increase in the Reynolds number, there is a consequent increase in the drag coefficient to approximately 0.7. For the Reynolds numbers above 4×10^6, the drag becomes independent of the Reynolds number.

Figure 10.26 shows an approximate relationship between C_d and R_e for an infinitely long cylinder.

It needs to be borne in mind that C_d is not an absolute constant for a given shape. It will vary with the speed of the fluid flow being a function of the Reynolds number (R_e). As an example, the C_d of a smooth sphere will vary from a high value for a laminar flow to 0.47 when in a turbulent fluid flow. Figure 10.27 shows the variation in the C_d factor for a sphere with respect to the increasing Reynolds number.

An empirical formula that covers the Reynolds number in the range $R_e = 0.2$ to 10^5 is as follows:

$$C_d = \frac{24}{R_e} + \frac{6}{1 + \sqrt{R_e}} + 0.4 \qquad (10.77)$$

EXAMPLE 10.16

A 40.0 mm diameter sphere is suspended in a flowing fluid having a density of 750 kg/m³ and a dynamic viscosity of 50 cP. The fluid is flowing at 0.6 m/s. (1 cP = 0.001 N · s/m².)

Calculate the drag force acting on the sphere.

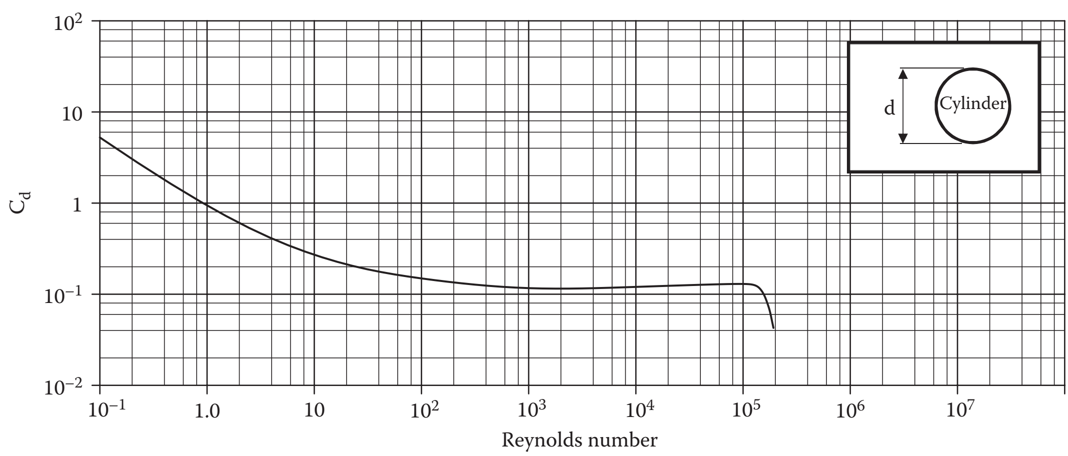


FIGURE 10.26 Drag factor (C_d) for a cylinder.

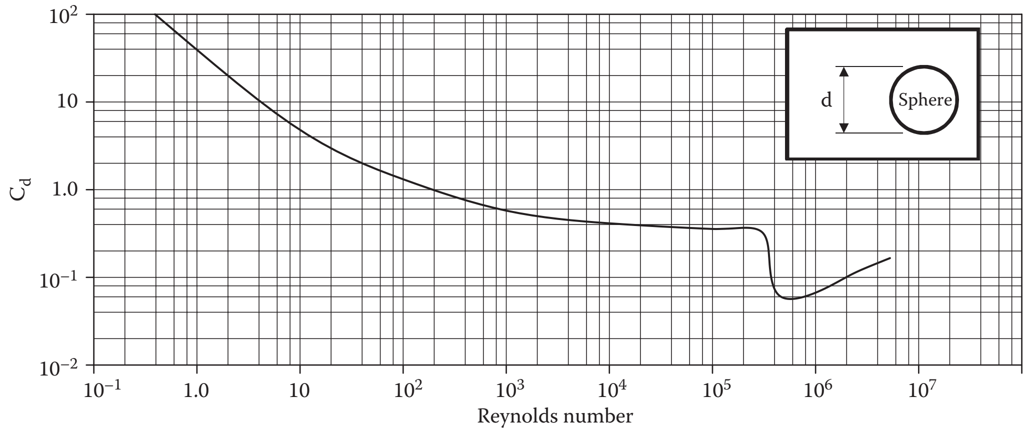


FIGURE 10.27 Drag factor (C_d) for a sphere.

Solution

$$R_e = \frac{\rho u d}{\mu}$$
$$= \frac{750 \times 0.6 \times 0.04}{0.05}$$
$$= 360$$

From Equation 10.77:

$$C_d = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$$
$$= \frac{24}{360} + \frac{6}{\sqrt{360}} + 0.4$$
$$C_d = 0.7671$$

Now

$$\text{Projected area} = \frac{\pi d^2}{4}$$
$$= \frac{\pi \times 0.04^2}{4}$$
$$= 1.2566 \times 10^{-3}\ \text{m}^2$$
$$C_d = \frac{2R}{\rho u^2 \times \text{projected area}}$$

Rearranging the equation to solve for 'R':

$$R = \frac{C_d \rho u^2 \cdot A}{2}$$
$$= \frac{0.8 \times 750 \times 0.6^2 \times 1.2566 \times 10^{-3}}{2}$$
$$= 0.136\ \text{N (Ans)}$$

10.7 PROPERTIES OF WATER

Water is described as a colourless, transparent, odourless, and tasteless compound of oxygen and hydrogen and is the most abundant compound found on the planet. It covers approximately 70% of the Earth's surface. Water exists in all three states: solid, liquid and gas. It is in dynamic equilibrium between liquid and gas states.

Table 10.6 lists the chemical and physical properties of water and Table 10.7 gives the properties of water over a range of varying temperatures and saturation pressures. Except where noted otherwise, data are given for water in their standard state (at 25°C, 100 kPa).

TABLE 10.6
Chemical and Physical Properties of Water

Some Common Thermal Properties of Water	
Maximum density at 4°C	1000 kg/m^3
Specific weight at 4°C	9.807 kN/m^3
Freezing temperature	0°C
Boiling temperature	100°C
Latent heat of melting	334 kJ/kg
Latent heat of evaporation	2.270 kJ/kg
Critical temperature	380–386°C
Critical pressure	22.1 MPa (MN/m^2)
Specific heat—water	4.187 kJ/kg K
Specific heat—ice	2.108 kJ/kg K
Specific heat water vaporisation	2.996 kJ/kg K
Thermal expansion from 4°C to 100°C	0.42×10^{-3} K^{-1}
Bulk modulus elasticity	2.15×109 Pa (N/m^2)
Identifiers	
CAS number	7732-18-5
PubChem	962
ChemSpider	937
UNII	059QF0KO0R
ChEBI	CHEBI:15377
ChEMBL	CHEMBL1098659
RTECS number	
Properties	
Molecular formula	H_2O
Molar mass	18.01528(33) g/mol
Density	1000 kg/m^3
Melting point	0°C, 32°F, 273.15 K
Boiling point	99.98°C, 211.97°F, 373.13 K
Acidity (pK_a)	15.74~35–36
Basicity (pK_b)	15.74
Refractive index (nD)	1.3330
Viscosity	0.001 Pa s at 20°C
Structure	
Crystal structure	Hexagonal
Molecular shape	Bent
Dipole moment	1.85 D

TABLE 10.7
Properties of Water at Varying Temperatures and Saturation Pressures

Temp	Saturation Pressure	Density	Density	Specific Volume	Specific Volume	Viscosity	Viscosity
t	p_{sat} Pressure	ρ_{lig} Liquid	ρ_{vap} Vapour	v_{liq} Liquid	v_{vap} Vapour	μ_{lig} Liquid	μ_{vap} Vapour
°C	MPa	kg/m³	kg/m³	m³/kg	m³/kg	μPa · s	μPa · s
0.01	0.0006112	999.8	0.004851	0.001	206.1	1792	9.216
10	0.001228	999.7	0.009407	0.001	106.3	1306	9.461
20	0.002339	998.2	0.01731	0.001002	57.76	1002	9.727
30	0.004247	995.6	0.03042	0.001004	32.88	797.4	10.01
40	0.007385	992.2	0.05124	0.001008	19.52	653.0	10.31
50	0.01235	988.0	0.08315	0.001012	12.03	546.8	10.62
60	0.01995	983.2	0.1304	0.001017	7.667	466.4	10.93
70	0.0312	977.7	0.1984	0.001023	5.040	403.9	11.26
80	0.4741	971.8	0.2937	0.001029	3.405	354.3	11.59
90	0.07018	965.3	0.4239	0.001036	2.359	314.4	11.93
100	0.1014	958.3	0.5982	0.001043	1.672	281.7	12.27
110	0.1434	950.9	0.8269	0.001052	1.209	254.7	12.61
120	0.1987	943.1	1.1220	0.001060	0.8912	232.1	12.96
130	0.2703	934.8	1.4970	0.001070	0.6680	212.9	13.30
140	0.3615	926.1	1.9670	0.001080	0.5085	196.5	13.65
150	0.4762	917.0	2.548	0.001091	0.3925	182.5	13.99
160	0.6182	907.4	3.260	0.001102	0.3068	170.2	14.34
170	0.7922	897.5	4.122	0.001114	0.2426	159.6	14.68
180	1.003	887.0	5.159	0.001127	0.1938	150.1	15.03
190	1.255	876.1	6.395	0.001141	0.1564	141.8	15.37
200	1.555	864.7	7.861	0.001157	0.1272	134.3	25.71
210	1.908	852.7	9.588	0.001173	0.1043	127.6	16.06
220	2.320	840.2	11.62	0.001190	0.6609	121.5	16.41
230	2.797	827.1	13.99	0.001209	0.0715	116.0	16.76
240	3.347	813.4	16.75	0.001229	0.0597	110.9	17.12
250	3.976	798.9	19.97	0.001252	0.05008	106.1	17.49
260	4.692	783.6	23.71	0.001276	0.04217	101.7	17.88
270	5.503	767.5	28.07	0.001303	0.03562	97.50	18.28
280	6.417	750.3	33.16	0.001333	0.03015	93.51	18.70
290	7.442	731.9	39.13	0.001366	0.02555	89.66	19.15
300	8.588	712.1	46.17	0.001404	0.02166	85.90	19.65
310	9.865	690.7	54.54	0.001448	0.01833	82.17	20.21
320	11.280	667.1	64.64	0.001499	0.01547	78.41	20.85
330	12.860	640.8	77.05	0.001561	0.01296	74.54	21.61
340	14.600	610.7	92.76	0.001638	0.01078	70.43	22.55
350	16.530	574.7	113.6	0.001740	0.008802	65.88	23.82
360	18.67	527.6	143.9	0.001895	0.006949	60.33	25.72

In line with a number of substances, water can take three basic forms:

Solid: The solid phase of water is known as ice and commonly takes the structure of hard, amalgamated crystals, such as ice cubes or loosely accumulated granulated crystals such as snow.

Liquid: Water is mostly in the liquid phase at the Earth's surface at normal temperature and pressure.

Gas: When heat is applied to water, it transforms to a gaseous phase and is known as water vapour or steam, which is essentially minute droplets suspended in the air and forming a transparent cloud.

With an increase in temperature and pressure, the water enters a fourth state which is a supercritical fluid. This state is much less common than the other three; it only very rarely occurs in nature and happens in deep water hydrothermal vents where the water is heated to the supercritical temperature by the volcanic magna and the critical pressure is achieved by the crushing weight of the ocean at the extreme depths where the vents are located.

In power generation water boilers, where a supercritical phase can be achieved, the boiler will be operating at its most efficient rate where the water is immediately converted into supercritical steam without the vapour phase occurring.

10.7.1 Specific Heat Capacity of Water

Water has a very high specific heat capacity (the second highest after ammonia) together with a high heat of vaporisation (2257 kJ/kg at normal boiling point). This is a result of extensive bonding of hydrogen between its molecules.

The specific heat is defined as the heat required to raise the temperature of a 1.0 kg of a substance by 1.0°C (K). Its units are J/kg K (or J/kg °C).

The designation for specific heat at constant pressure is 'c'.

When heat is transferred to a liquid, the temperature of the liquid will rise and is directly proportional to the heat transferred (Q):

$$Q = mc\Delta T,$$

where

m is the mass (kg)
c is the specific heat index
ΔT is the rise in temperature (°C or K)

The specific heat 'c' is reasonably constant but will change significantly when the pressure or temperature change is very large.

EXAMPLE 10.17

Calculate the rise in temperature of 5 kg of water when 84 kJ of heat energy is applied.

Solution

Specific heat capacity of water = 4.187 kJ/kg K.
Heat energy input:

$$\begin{aligned} Q &= mc\Delta T \\ \Delta T &= \frac{Q}{mc} \\ &= \frac{84 \times 10^3}{5 \times 4.187 \times 10^3} \\ &= 4.012\,\text{K (Ans)} \end{aligned}$$

10.7.2 Enthalpy of Fusion

The enthalpy of fusion or heat of fusion is the change in enthalpy resulting from heating a given quantity of a substance to change its state from a solid to a liquid. The temperature at which this occurs is called the melting point.

The 'enthalpy' of fusion is a latent heat, because during the change in state, the introduction of heat cannot be observed as a temperature change, as the temperature remains constant during the process. The latent heat of fusion is the enthalpy change of any amount of substance when it melts. When the heat of fusion is referenced to a unit of mass, it is usually called the specific heat of fusion, while the molar heat of fusion refers to the enthalpy change per amount of substance in moles.

The liquid phase has a higher internal energy than the solid phase. This means energy must be supplied to a solid in order to melt it and energy is released from a liquid when it freezes, because the molecules in the liquid experience weaker intermolecular forces and have a larger potential energy.

When liquid water is cooled, its temperature falls steadily until it drops just below the freezing point at 0°C. The temperature then remains constant at the freezing point while the water crystallises. Once the water is completely frozen, its temperature continues to fall.

The specific enthalpy of fusion of water is 333.55 kJ/kg at 0°C. Of common substances, only that of ammonia is higher.

10.7.3 Enthalpy of Vaporisation

The enthalpy of vaporisation, H_V, also known as the heat of vaporisation or heat of evaporation, is the energy required to transform a given quantity of a substance from a liquid into a gas at a given pressure (often atmospheric pressure).

It is often measured at the normal boiling point of a substance; although tabulated values are usually corrected to 298 K, the correction is often smaller than the uncertainty in the measured value.

The heat of vaporisation is temperature dependent (Table 10.8), though a constant heat of vaporisation can be assumed for small temperature ranges and for $Tr \ll 1.0$. The heat of

TABLE 10.8
Heat of Vaporisation

Temperature	Hv (kJ/mol (°C))
0	45.054
25	43.99
40	43.35
60	42.482
80	41.585
100	40.657
120	39.684
140	38.643
160	37.518
180	36.304
200	34.962
220	33.468
240	31.809
260	29.93
280	27.795
300	25.3
320	22.297
340	18.502
360	12.966
374	2.066

TABLE 10.9
Constant-Pressure Heat Capacity

Temperature (°C)	Cp (J/(g K) at 100 kPa)
0	4.2176
10	4.1921
20	4.1818
25	4.1814
30	4.1784
40	4.1785
50	4.1806
60	4.1843
70	4.1895
80	4.1963
90	4.205
100	4.2159

vaporisation diminishes with increasing temperature and it vanishes completely at the critical temperature ($T_r = 1$) because above the critical temperature, the liquid and vapour phases no longer co-exist.

Table 10.9 gives the constant pressure heat capacity (C_p) for a range of temperatures in J/(g · K) at a pressure of 100 kPa.

10.8 CHANNEL FLOW

Open-channel flow, a branch of hydraulics, is a type of liquid flow within a conduit with a free surface, known as a channel. The other type of flow within a conduit is pipe flow. These two types of flow are similar in many ways, but differ in one important respect: the free surface. Open-channel flow has a free surface, whereas most pipe flows do not.

Open channels are found in a variety of situations; namely, drainage systems and irrigation systems. They are also found when diverting water from a natural reservoir in nearby hills to a water treatment plant for supplying clean drinkable water to towns and cities.

10.8.1 Channel Flow

In the above examples, the water flows down an open channel with its top surface open to the air. Engineers have to ensure that these channels have sufficient capacity to carry the anticipated flow of water.

Figure 10.28a–d shows various cross-sections of typical open channel shapes. Types of open channel flow:

- Steady flow—when the flow rate (Q) does not change with time.
- Uniform flow—when the depth of fluid does not change for a selected length or section of the channel.
- Uniform steady flow—when the discharge does not change with time and the depth remains constant for a selected section, cross-section should remain constant.
- Varied steady flow—when the depth changes but the discharge remains the same.
- Varied unsteady flow—when both the depth and discharge changes along the length of the channel being measured.

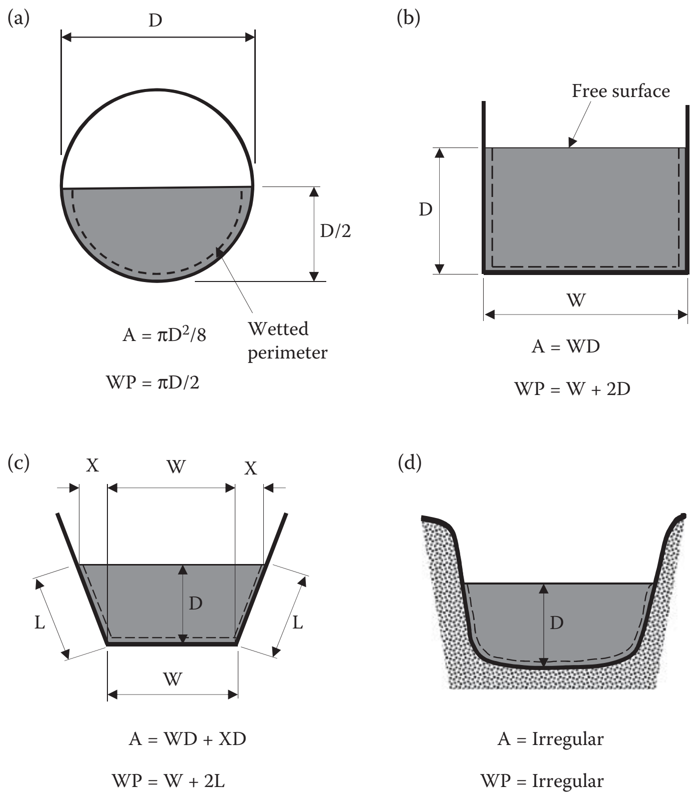


FIGURE 10.28 Typical open channels. (a) Circular pipe running half full, (b) rectangular channel, (c) trapezoidal channel and (d) natural channel.

- Rapidly varying flow—depth change is rapid, and the cross-section remains constant.
- Gradually varying flow—the depth change is gradual with a constant cross-section remaining constant.

10.8.2 Hydraulic Radius

A parameter that is used to express the ratio of the flow cross-sectional area (A) of the channel and the wetted perimeter (WP):

$$R = \frac{A}{WP} \tag{10.78}$$

10.8.3 Flow Rate

Flow rate (Q) is the rate at which the fluid, in terms of volume, flows along a channel. Its units are m^3/s.

If it is assumed that the flow in the channel is uniform, then the flow rate is a function of

- The area of the channel (m^2)
- The hydraulic radius (m)

- The slope of the channel (in the direction of the flow)
- The roughness of the channel material that is in direct contact with the fluid

The above assumptions can be represented using the 'Manning equation':

$$Q = \frac{A}{n} \times R^{2/3} \times S^{1/2} \tag{10.79}$$

where

A represents cross-sectional area of the channel (m^2)
R represents hydraulic radius (m)
S represents channel slope in the direction of flow (m/m)
n represents Manning coefficient of roughness

10.8.4 Roughness

Roughness is dependent upon the nature of the channel material. Some values of Manning's values for 'n' are given in Table 10.10. From the table, it is seen that most materials have a range of values. When considering concrete as a material for construction, the smoothest concrete with a high class finish has a normal value of approximately 0.011, with a range of 0.010–0.013. Laboratory channels with glass side, used for research purposes and modelling, have a value of 0.010.

EXAMPLE 10.18

Estimate the normal discharge for a 200 mm inside diameter pipe, lined with a common clay tile, running half full. The slope drops 1.0 m over 1000 m.

Solution

$$A = \frac{\pi D^2}{8} = \frac{\pi \times (200 \times 10^{-3})^2}{8} = 0.0157\ m^2$$

TABLE 10.10
Various Values of 'n' for the Most Common Channel Surfaces

Channel Surface Description	n
Glass, copper, plastic or other smooth surface	0.010
Smooth, unpainted steel, planed wood	0.012
Painted steel or coated cast iron	0.013
Smooth asphalt, common clay drainage tile, trowel finished concrete, glazed brick	0.013
Uncoated cast iron, black wrought iron pipe, vitrified clay sewer pipe	0.014
Brick in cement mortar, float finished concrete, concrete pipe	0.015
Formed unfinished concrete, spiral steel pipe	0.017
Smooth earth	0.018
Clean excavated earth	0.022
Corrugated metal storm drain	0.024
Natural channel with stones and weeds	0.030
Natural channel with light brush	0.050
Natural channel with tall grasses and reeds	0.060
Natural channel with heavy brush	0.100

$$WP = \frac{\pi D}{2} = \frac{\pi \times 200 \times 10^{-3}}{2} = 0.3142\,m$$

$$R = \frac{A}{WP} = \frac{0.0157}{0.3142} = 0.050\,m$$

From Table 10.10, Manning's coefficient for clay tile is 0.013.
Substituting these values into Equation 10.79:

$$Q = \frac{A}{n} \times R^{2/3} \times S^{1/2}$$

$$= \frac{0.0157}{0.013} \times 0.05^{2/3} \times 0.001^{1/2}$$

$$Q = 5.183 \times 10^{-3}\,m^3/s\ (Ans)$$

EXAMPLE 10.19

Calculate the slope of the trapezoidal channel shown in Figure 10.30 when the flow rate is 1.42 m^3/s. It is formed from unfinished concrete.

Solution

$$A = WD + XD$$
$$= (2.0 \times 0.50) + (0.5 \times 0.5)$$
$$= 1.25\ m^2$$

$$WP = W + 2L$$
$$= 2.0 + (2 \times 0.7071)$$
$$= 3.414\ m$$

$$R = \frac{A}{WP}$$
$$= \frac{1.25}{3.414}$$
$$= 0.366\,m$$

Substitute values into Equation 10.79:

$$Q = \frac{A}{n} \times R^{2/3} \times S^{1/2}$$

$$1.42 = \frac{1.25}{0.017} \times 0.366^{2/3} \times S^{1/2}$$

Rearranging:

$$S = \left(\frac{n \cdot Q}{A \cdot R^{2/3}}\right)^2$$

$$= 1.445\,m\ per1000\,m\ (Ans)$$

10.9 ORIFICE PLATE

An orifice plate is an artificial obstruction placed in a pipeline that is used for the measurement of flow rate. Volumetric or mass flow rates may then be determined, this being dependent upon the calculation method that is associated with the specific type of orifice plate used.

Bernoulli's principle is used in the measurement of the fluid flow, where there is a relationship between the pressure of the fluid and its velocity, that is, when the velocity increases there is a corresponding drop in the fluid pressure and vice versa.

10.9.1 Description

An orifice plate is a thin plate having a hole carefully machined in the centre of the plate. The plate is fixed in the pipeline and is mounted between two coupling flanges in the pipe. The fluid is forced to go through the small orifice; there is a point of maximum convergence that occurs a short distance downstream of the plate. This convergence is known as the vena contracta (see Figure 10.29). The velocity of the fluid is increased when going through the orifice and there is a corresponding reduction in the fluid pressure. Beyond the vena contracta, the fluid expands back to the pipeline diameter and the velocity and pressure return to their original values. Pressure tappings are provided, one upstream of the plate and the other just downstream.

There are three locations for the tappings:

1. *Flange location:* The tap locations are 25 mm upstream and 24 mm downstream from the face of the orifice plate.
2. *Vena contracta location:* The tap locations are one internal pipe diameter upstream and 0.3–0.8 pipe diameter downstream from the face of the orifice plate.
3. *Pipe location:* The tap location 2.5 internal pipe diameter upstream, and 8 pipe diameter downstream from the orifice plate.

It should be noted here that an orifice plate will only work well with a fully developed flow profile. This is achieved by using a long upstream pipe usually between 20 and 40 pipe diameter before the orifice plate depending upon the Reynolds number or by the use of a flow conditioner. Orifice plates are inexpensive and small but the pressure drop is not so well recovered as well as a venturi nozzle. If there is sufficient space available, a venturi nozzle is more efficient than an orifice plate.

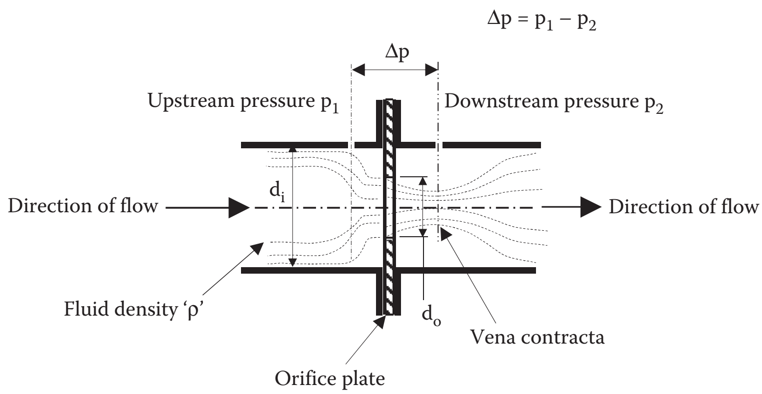


FIGURE 10.29 Orifice plate.

10.9.2 Measurement

When the fluid flows through the orifice plate, there are frictional energy losses which together with the contraction in the diameter of the flow affect the value of 'C_d' (coefficient of discharge). The shape of the orifice lip is also important; this may be either sharp or flat.

If the Bernoulli equation is applied between positions 1 and 2 in Figure 10.29, the following is obtained:

$$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{u_2^2}{2g} \tag{10.80}$$

Neglecting u_1^2:

$$h_1 = h_2 + \frac{u_2^2}{2g} \tag{10.81}$$

$$2g(h_1 - h_2) = u_2^2$$

$$u_2 = \sqrt{2g(h_1 - h_2)}$$

or

$$u_2 = \sqrt{\frac{2\Delta p}{\rho}} \tag{10.82}$$

now

$$Q = u_2 \cdot A_o$$

Introducing 'C_d' to allow for friction and other factors:

$$Q = C_d \cdot A_o \sqrt{2g(h_1 - h_2)}$$

which can be expressed as

$$Q = C_d A_o \sqrt{\frac{2\Delta p}{\rho}} \tag{10.83}$$

where

Q represents volumetric flow rate (m/s^3)
A_o represents area of orifice diameter (m)
Δp represents differential pressure (Pa)
ρ represents fluid density (kg/m^3)

The majority of commercial orifice plate calculations only consider the diameter of the orifice to calculate the volumetric flow rate.

EXAMPLE 10.20

A horizontal pipe with an internal diameter of 25 mm is fitted with an orifice plate having a 20 mm diameter hole. The differential pressure across the plate is measured at 750 kPa. It can be assumed that the coefficient of discharge is 0.64 and the density of water is 1000 kg/m^3.

Estimate the volumetric flow rate in m^3/s.

Solution

$$A_o = \frac{\pi d_o^2}{4} = \frac{\pi \times 0.02^2}{4} = 0.314 \times 10^{-3} m^2$$

$$Q = C_d A_o \sqrt{\frac{2\Delta p}{\rho}}$$

$$= 0.64 \times 0.314 \times 10^{-3} \times \sqrt{\frac{2 \times 750{,}000}{1000}}$$

$$= 0.007783\, m^3/s \text{ (Ans)}$$

10.10 FLUID MACHINES

A fluid machine is a device that converts the kinetic or potential energy held in a fluid into mechanical energy, or it is reversible and converts mechanical energy into fluid energy.

Essentially, there are two main types of fluid machines, namely:

1. Positive displacement machines
2. Rotodynamic machines

10.10.1 Positive Displacement Machines

These come in three forms:

1. Single rotor
2. Double rotor
3. Reciprocating piston

10.10.1.1 Single Rotor

Sliding vane: In this design the vanes are moved by a rotor, drawing fluid into and then forcing the fluid out from the pumping chamber that is formed within the body of the pump (see Figure 10.30). Carefully selected vane materials make these pumps well-suited for handling low-viscosity, non-lubricating liquids including solvents, fuel oils, petrol and liquidified gas. Fluid viscosities range from 0.5 cSt to 220,000 cSt.

Axial piston: This is a variation in the piston pump described above; here the individual pistons are arranged in a circular manner within a cylinder block and are driven by a cam plate fitted on the drive shaft. In this type of pump, the fluid is drawn in and forced out as the inclined cam plate is rotated about the centreline of the pump. This type of pump has the advantage that the flow can be variable; the angle of the cam plate is varied from a zero angle where there will be no piston displacement, up to a maximum angle where the pistons will have a maximum stroke. These pumps can also be supplied with a fixed cam plate.

Axial piston pumps have a relatively low flow rate of approximately 70 m^3/h (300 galls/min) but are capable of generating pressures up to 25 MPa (3626 lbf/in^2). Figure 10.31 depicts an axial piston pump.

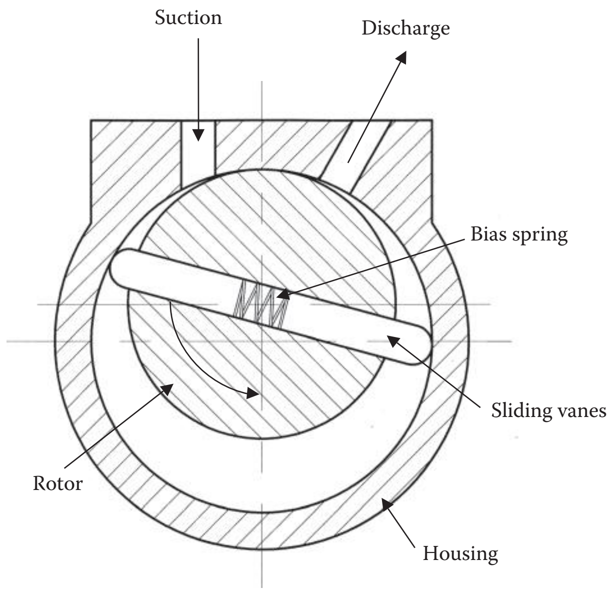


FIGURE 10.30 Sliding vane pump.

Reciprocating piston pump: Fluid is drawn in and out of the pump chamber by a piston that reciprocates within a cylinder. The flow of the fluid is controlled by port valves attached to the cylinder (see Figure 10.32).

Peristaltic pump: A peristaltic pump is a type of positive displacement pump used for pumping a variety of fluids. The fluid is contained within a flexible tube fitted inside a circular pump casing (though linear peristaltic pumps have been made). A rotor with a number of 'rollers', 'shoes', 'wipers' or 'lobes' attached to the external circumference of the rotor compresses the flexible tube. As the rotor turns, the part of the tube under compression is pinched closed (or 'occludes'), thus forcing the fluid to be pumped to move through the tube (Figure 10.33). Additionally, as the tube opens to its natural state after passing the cam (restitution or resilience), fluid flow is induced into the pump. This process is called peristalsis. Typically, there will be two or more rollers or wipers, occluding the tube, trapping between them a body of fluid. The body of fluid is then transported, at ambient pressure,

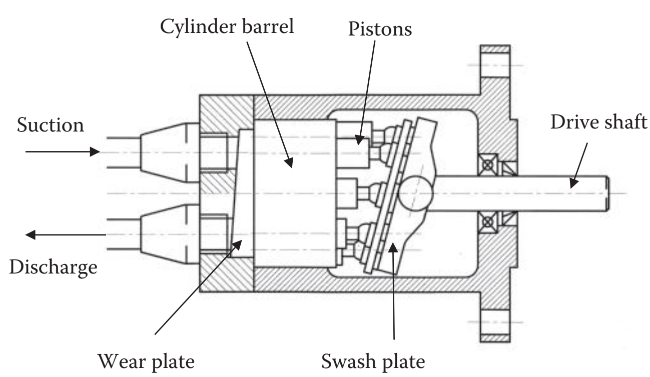


FIGURE 10.31 Axial piston pump.

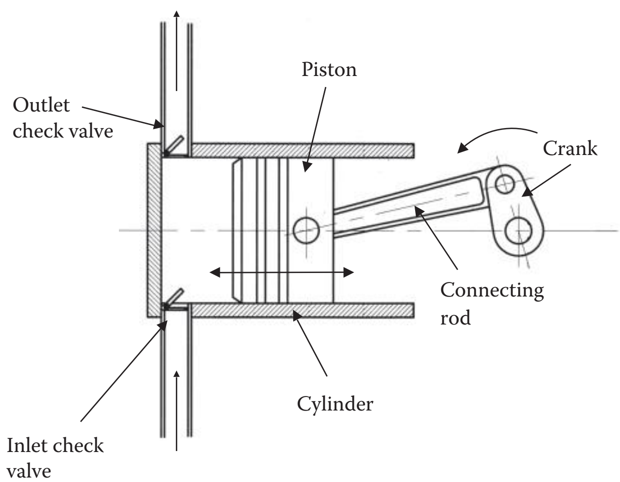


FIGURE 10.32 Reciprocating pump.

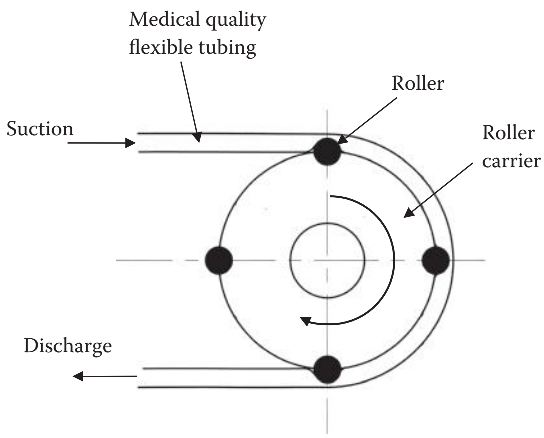


FIGURE 10.33 Peristaltic pump.

towards the pump outlet. These pumps are able to pump fluids with solids in suspension and low-to-medium viscosity levels. Peristaltic pumps are available with flow rates up to 1.4 m^3/min (307 gallons/min) with differential pressures up to 1.6 MPa (232 lbf/in^2).

Smaller models generally operate at speeds up to 200 rev/min and the larger models are limited to speeds below 100 rev/min.

10.10.1.2 Double Rotor

Circumferential pump: This type of pump as shown in Figure 10.34 is similar to the lobe pump but here there are only two 'teeth' per shaft and the fluid is carried around in the space between the piston surfaces and the fluid chamber. Like the lobe pump, there is no sealing contact between the piston surfaces, clearances between the pistons are maintained by external timing gears. These pumps have proven to be very reliable. As the circumferential pump rotates, the expanding volume draws the fluid in and is then forced out the discharge port by the reducing volume on the discharge side.

Circumferential pumps are available in a range of sizes capable of flow rates up to 2.3 m^3/min (506 galls/min) and discharge pressures up to 3.1 MPa (450 lbf/in^2) covering

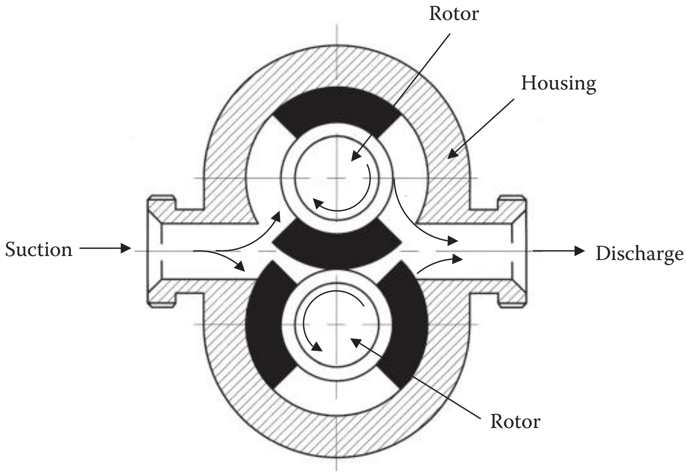


FIGURE 10.34 Circumferential piston.

a viscosity range from 50 to 1,000,000 cSt. They are able to pump any product that can be moved being able to handle rather large solids and shear-sensitive fluids. They are also suitable to run dry for extended periods of time.

Gear pump: (Figure 10.35). Fluid is carried between the gear teeth within a tight fitting circular cavity and is expelled by the meshing teeth. The inner surface of the pump housing provides a continuous sealing and either rotor is capable of driving the other.

Lobe: Lobe pumps are similar to external gear pumps in operation in that the fluid flows around the interior of the casing (Figure 10.36). There are three lobes per shaft leaving a cavity between the lobe features. Unlike external gear pumps, however, the lobes do not make contact. Lobe contact is prevented by external timing gears located in the gearbox. Pump shaft support bearings are located within the gearbox, and since the bearings are out of the pumped liquid, the pressure is limited by bearing location and shaft deflection. As the lobes come out of mesh, they create expanding volume on the inlet side of the pump. The liquid flows into the cavity and is trapped by the lobes as they rotate. Liquid travels around the interior of the

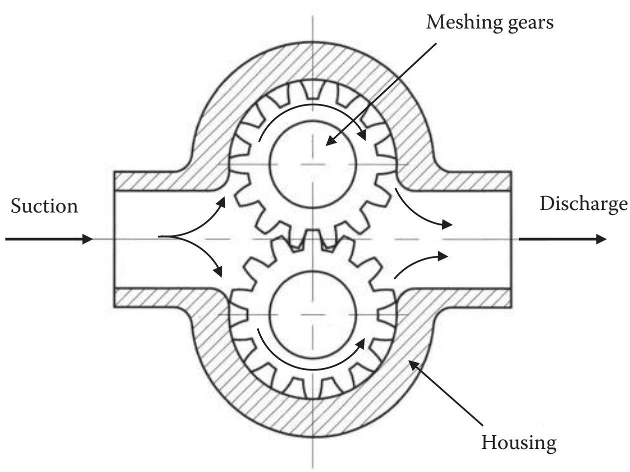


FIGURE 10.35 Gear pump.

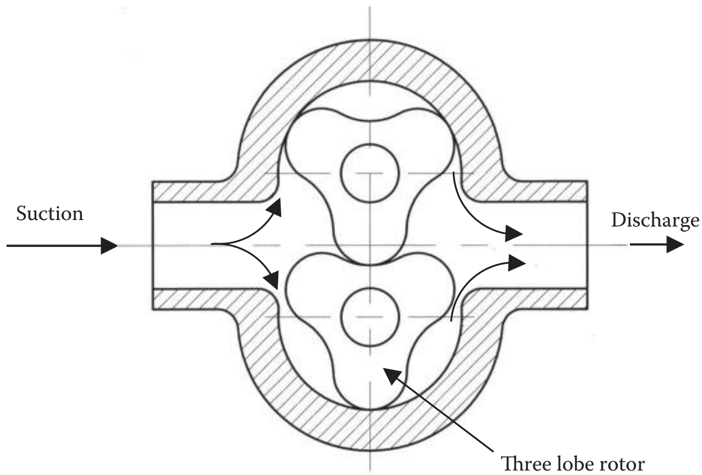


FIGURE 10.36 Lobe pump.

casing in the pockets between the lobes and the casing—it does not pass between the lobes. Finally, the meshing of the lobes forces the liquid through the outlet port under pressure.

Single-screw pump: The fluid is carried between the rotor screw threads which are a close fit within a cylinder block. The Archimedes screw is one such example of this machine (see Figure 10.37). They can handle fluids containing abrasive and solid particles up to 90 mm (3.50 ins) and can also handle multiphase fluids containing up to 99% gas. These types of pumps are self-priming and can lift fluids up to 8.5 m (28 ft) and are able to handle any fluid that is compatible with the materials of construction.

Double-screw pump: Double-screw pumps have twin external screw rotors that intermesh as shown in Figure 10.38. The fluid enters at the central inlet and then splits axially into two end sections and as the rotating screws intermesh, chambers are formed and convey the fluid to the end receiving chambers formed in the housing and returned to a separate central port.

As with the single screw pumps, they can handle a wide range of multiphase fluids and are often found in oil production/pipeline applications.

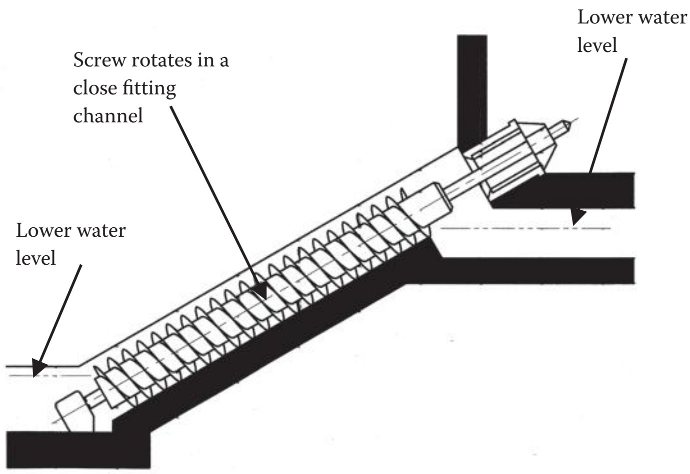


FIGURE 10.37 Single-screw pump.

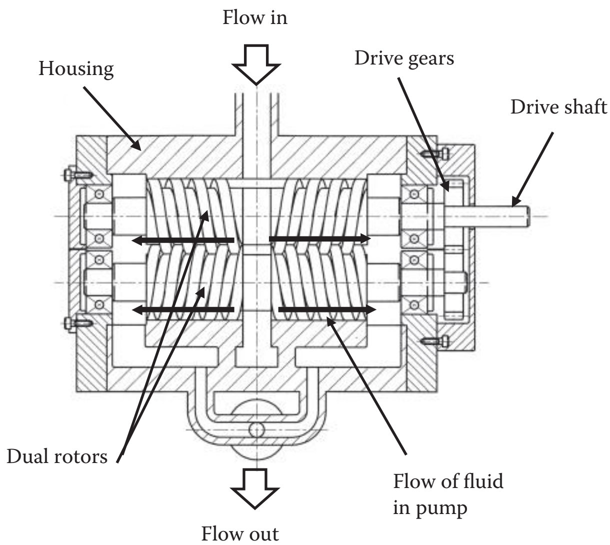


FIGURE 10.38 Double-screw pump.

Rotodynamic machines: Within this class of machines, there are three basic pump categories:

1. Radial flow
2. Mixed flow
3. Axial flow

Radial flow pumps—(Figure 10.39): Often simply referred to as centrifugal pumps. The fluid enters along the axial plane, is accelerated by the impeller and exits at right angles to the shaft (radially). Radial-flow pumps operate at higher pressures and lower flow rates than axial and mixed-flow pumps.

Mixed-flow pumps—(Figure 10.40): Mixed-flow pumps function as a compromise between radial and axial-flow pumps. The fluid experiences both radial acceleration and lift and exits the impeller somewhere between 0° and 90° from the axial

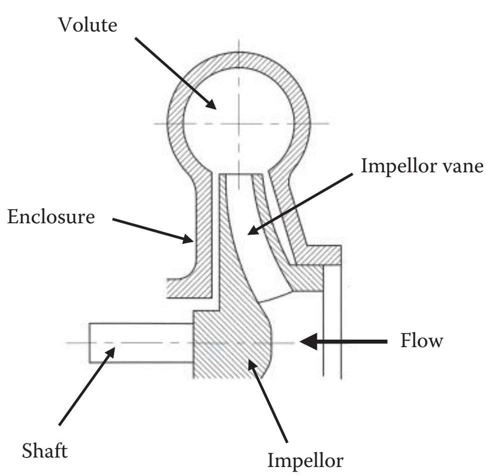


FIGURE 10.39 Radial flow compressor.

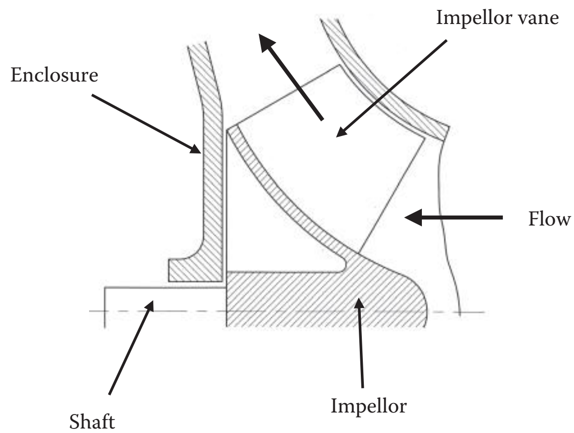


FIGURE 10.40 Mixed flow compressor.

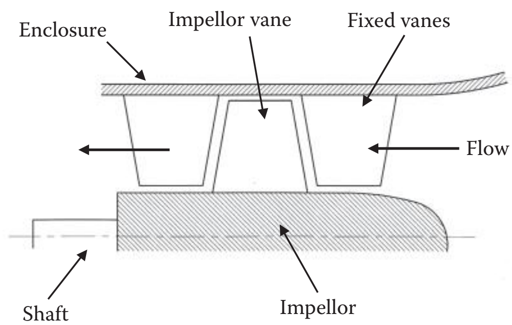


FIGURE 10.41 Axial flow compressor.

direction. As a consequence, mixed-flow pumps operate at higher pressures than axial-flow pumps while delivering higher discharges than radial-flow pumps. The exit angle of the flow dictates the pressure head-discharge characteristic in relation to the radial and mixed flow.

Axial flow pumps—(Figure 10.41): Axial-flow pumps differ from radial flow pumps in that the fluid enters and exits along the same direction parallel to the rotating shaft. The fluid is not accelerated but instead 'lifted' by the action of the impeller. They may be likened to a propeller spinning in a length of a tube. Axial-flow pumps operate at much lower pressures and higher flow rates than radial-flow pumps.

11 Introduction to Linkages

11.1 INTRODUCTION

The use of linkages may be found in many walks of life. An example in the home is of the rear suspension on a mountain bike. In the building industry, a further example will be of the hydraulically operated buckets fitted to wheeled loaders; in the aerospace industry, extensive use of linkages is found for extending and retracting the aircraft landing gear units including wing flaps and so on. In the automotive industry, the vehicle suspension systems rely heavily on the use of linkages for maintaining road wheel positions.

In manufacturing, robotic manipulators rely wholly on the science of linkages optimising their sections for strength, mass and deflections to enable them to carry heavy loads at high speeds from position to position.

The definition of a link is that of a rigid body having two or more pairing elements which connect it to other bodies with the prime purpose of transmitting force or motion.

This chapter will briefly cover planar, spatial and spherical mechanisms but focus on four bar linkages as these are the most commonly found and will give details of determining velocities and accelerations of linkages.

Linkages can be constructed from either open or closed chains or a combination of both. Each link in a chain is connected by a joint to one or more other links. These create a kinematic chain that can then be modelled using graphical methods in which the links are considered as paths and the joints are the vertices.

The movement of the ideal joint can be considered as a sub-group of the group of Euclidean displacements where the number of parameters within the sub-group is called the 'degrees of freedom' (DOF) of the joint.

One of the prime functions of a mechanical linkage is to transform a given input force or movement into a desired output force or movement. The ratio of the output force to the input force is referred to as the 'mechanical advantage' of the linkage, while the ratio of the input movement or speed to the output movement or speed is referred to as the 'velocity ratio'.

The velocity ratio and the mechanical advantage are defined so that they produce the same number in an ideal linkage.

By definition, a kinematic chain in which one of the links is fixed and held stationary is called a mechanism. Conversely, a linkage that is designed to be stationary is called a structure.

11.2 BRIEF HISTORY

The use of levers predates Western civilisation for moving heavy objects with only manpower available. Examples here include the construction of the pyramids during ancient Egyptian times. Archimedes studied the geometry of the lever and together with Hero of Alexandria became the prime sources of machine theory. This work lasted up to the fifteenth century when Leonardo da Vinci (1452–1519) in his Codex gave information about the use of linkages which he incorporated into his designs.

In the mid-1700s, several designs of steam engines were being proposed amongst them. Thomas Newcomen (1664–1729) designed his successful atmospheric steam engine, originally used in the tin mines of Cornwall for pumping water out of the mines. James Watt (1736–1819) recognised that the efficiency of these early engines could be significantly improved by using different cylinders for expansion and condensation of steam. His search for a linkage that could generate approximately straight lines inspired the mathematician J. J. Sylvester (1814–1897) who was an authority on the Peaucellier linkage which generated a straight line from a rotating crank. This work, in turn, inspired A. B. Kemp (1849–1922) who showed that linkages could be used for addition and multiplication and also for tracing a given algebraic curve.

Around this period, Charles Babbage (1791–1871) proposed his analytical and difference engine which made considerable use of cams and linkages, giving an indication of the degree of workmanship that was being achieved at this time.

In the late 1800s, F. Reuleaux (1829–1905), A. B. W Kennedy (1847–1928) and L. Burmester (1840–1927) formalised the analysis and synthesis of linkages using descriptive geometry. P. L. Chebyshev (1821–1894) developed analytical techniques for the study and design of linkages.

In recent years, G. N. Sandor (1913–1997) and F. Freudenstein (1926–2006), using the newly developed digital computer for solving the loop equations of a linkage, were able to establish the dimensions required to generate a desired function and begin the revolution for computer-aided-design of linkages. These techniques are now fundamental in the analysis of complex machine systems and the control of robotic manipulators.

By combining the computer's ability to compute the roots of polynomial equations, R. F. Kaufman united Freudenstein's techniques with the geometric methods of Reuleaux and Burmester to implement in *KINYN* an interactive computer program.

11.3 KINEMATIC DEFINITIONS

The following are some definitions that are commonly used in the analysis of linkages.

11.3.1 Kinematic Chain

An assemblage of links and bodies, together with joints, interconnected in such a way as to provide an output motion that is in response to an input motion.

11.3.2 Mechanism

A kinematic chain in which at least one link or body is fixed in relation to the frame of reference (in which it itself may be in motion).

11.3.3 Machine

A combination of resistant bodies was arranged in such a way as to compel the mechanical forces of nature to do work accompanied by determinate motions.

A kinematic link or bar is a rigid body or assembly which constitutes the parts of a mechanism and is the smallest element of the mechanism that transmits motion to other links. A rigid body is one which does not deform or change shape due to the application of a force. In a mechanism, three types of links, rigid link, flexible link and fluid link, are the most widely used.

11.3.4 DOF

DOF is defined as the number of input parameters which must be independently controlled in order to bring the mechanism into useful engineering purposes. It also defines the number of independent relative motions, both translational and rotational, that a pair can have. DOF = 6 – number of restraints.

To find the number of DOF for planar mechanisms, an equation referred to as Grubler's equation is used where

$$F = 3(n - 1) - 2j_1 - j_2 \tag{11.1}$$

where
F = mobility or number of DOF
n = number of links including frame
j_1 = joints with a single degree of freedom
j_2 = joints with two DOF
F > 0, results in a mechanism with 'F' DOF
F = 0, results in a statically determinate structure
F < 0, results in a statically indeterminate structure

11.3.5 Rigid Links

A rigid link possesses at least two nodes, where a node is an attachment point to other links or bodied via joints.

Links can be classified into the following three categories as depicted in Figure 11.1a–c.

1. Binary link: This is a link that has two nodes.
2. Ternary link: A link that forms three nodes.
3. Quaternary link: A link that has four nodes.

The ternary and quaternary links do not have any relative movement between the nodes within the link.

11.3.6 Order of a Link

The order of a link indicates the number of nodes per link as indicated above; there are binary (two), ternary (three) and quaternary (four) nodes. Links or bodies can be of any shape and are not restricted to those shown in Figure 11.1a–c.

Link order is equal to the number of nodes.

11.3.7 Joints

A joint is a connection between two or more links at their nodes allowing motion to occur between the links.

- A pivot is a joint that allows rotary motion at the joint.
- A slider is a joint that only allows a linear motion.

11.3.8 Kinematic Pairs

Kinematic pairs refer to the type of connection that exists between adjacent links. See Section 11.4 for a more detailed description of kinematic pairs and the definition in the differences between lower and higher pairs.

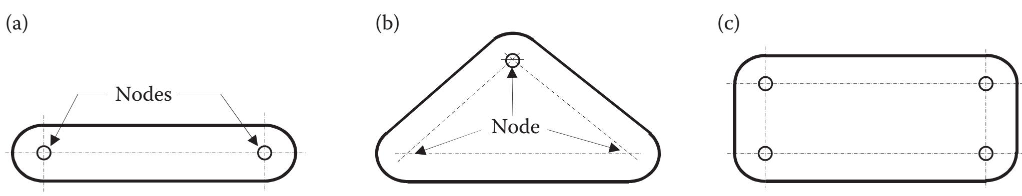


FIGURE 11.1 Classification of links. (a) Binary, (b) ternary and (c) quaternary links.

11.3.9 Mobility

This is the number of input parameters that have to be controlled independently to bring the mechanism to a set position.

See Section 11.6 for a fuller description.

11.4 KINEMATIC PAIRS

The part of a link that makes contact with another link is called an 'element', examples being either the bore of a cylinder or a gear tooth or a rotating connection with another link.

The connections between links are made by pairs of elements coming into contact; these connections are known as 'kinematic pairs'. The kinematic pairs fall into two categories:

1. Lower pairs
2. Higher pairs

Lower pairs can be defined as those having surface contact where the relative motion is either purely turning or sliding. The contact surfaces of a lower pair are complementary to each other, such as a journal or plain bearing or a piston within a cylinder. Lower pairs may be defined as those allowing one degree of freedom.

Higher pairs may be defined as those having line or point contact, such as gear teeth or a follower on a cam. The relative motion in higher pairs is a combination of sliding and turning. The contact surfaces are dissimilar. Kinematic pairs formed between gears, cams and followers and ball and roller bearings belong to the higher pair category. Higher pairs usually allow two DOFs.

11.4.1 Relative Motion between Kinematic Pairs

The nature of the relative motion between kinematic pairs can be further classified into the following types:

11.4.1.1 Lower Pairs

1. *Sliding pair:* A kinematic pair is said to be a 'sliding pair' when two links are so connected that one element is constrained to have a sliding motion relative to the element. Figure 11.2a illustrates a prismatic bar sliding in a rectangular hole. The connection between a piston within a cylinder of a slider-crank mechanism is a further example of a sliding pair.
2. *Revolute pair:* When two links are so connected that only a rotation motion is possible between them, the kinematic pair is called a 'turning pair'. A circular shaft that is retained by two collars as shown in Figure 11.2b depicts an example of a turning pair.
3. *Cylindrical pair:* Where two links are so connected that rotational motion together with linear motion is allowed, the kinematic pair is referred to as a cylindrical pair as shown in Figure 11.2c.
4. *Helical pair:* Figure 11.2d shows a screw pair which has rotation as well as translation but these two movements are related to each other. Therefore, a screw pair has one degree of freedom because the relative movement between the screw and nut can be expressed by a single coordinate 'θ' or 'x'. These two coordinates are related by

$$\frac{\Delta\theta}{2\pi} = \frac{\Delta x}{L}$$

where L is the lead of the screw.

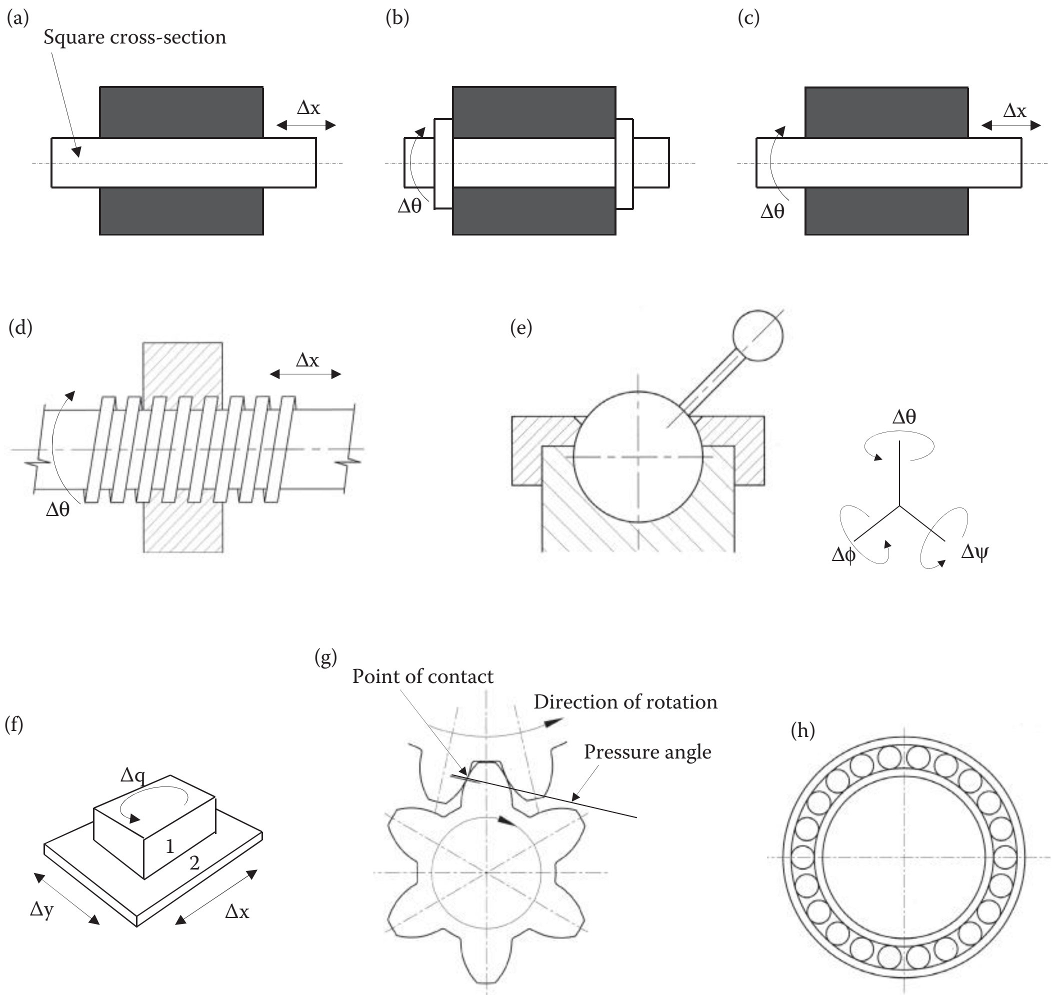


FIGURE 11.2 Kinematic pairs. (a) Prismatic (P) joint-1 DOF, (b) revolute (R) joint-1 DOF, (c) cyclindrical (C) joint-2 DOF, (d) helical (H) joint-1 DOF. joint, (e) spherical (S) joint-3 DOF. joint, (f) planar (F) joint-3 DOF, (g) gear pair and (h) roller bearing.

5. *Spherical pair:* When one link is in the form of a sphere that is allowed to turn within a fixed link, such a pair is referred to as a 'spherical pair', such an example being a ball and socket joint (Figure 11.2e).
6. *Planar pair:* Where the relative motion between items 1 and 2 can be described by x and y coordinates in an x–y plane. The x and y coordinates describe relative translation and 'θ' describes relative rotation about the z-axis. This pair has three DOFs. This is shown in Figure 11.2f.

11.4.1.2 Higher Pairs

7. *Gear pair:* The contact between the involute curves that form the meshing teeth of two gears is known as a gear pair. The contact between the teeth is considered a point contact. Other examples in this category include cams and followers. Figure 11.2g shows a gear pair.

8. *Rolling pair:* A pair of elements having a rolling motion relative to each other is called a 'rolling pair', where the contact with its mating part is a line contact. Examples are a ball or roller within a cage as shown in Figure 11.2h and a pulley in a belt drive.

11.4.2 Nature of Kinematic Constraints

Subject to the nature of the mechanical constraints, kinematic pairs can be classified into the following two categories.

11.4.3 Closed Pair

When two links that form a kinematic pair (Figure 11.3) are restrained mechanically, such a pair is called a 'closed pair'. Examples include screw pairs and turning pairs.

11.4.4 Open Pair

Two links that form a pair (Figure 11.4) that is held together by means of an external force such as gravity or spring force will constitute an 'open pair'. The cam follower used in an IC engine valve operating mechanism is an example of an open pair.

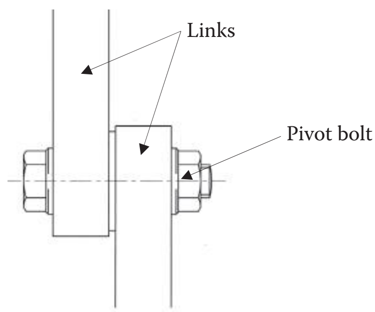


FIGURE 11.3 Closed pairs.

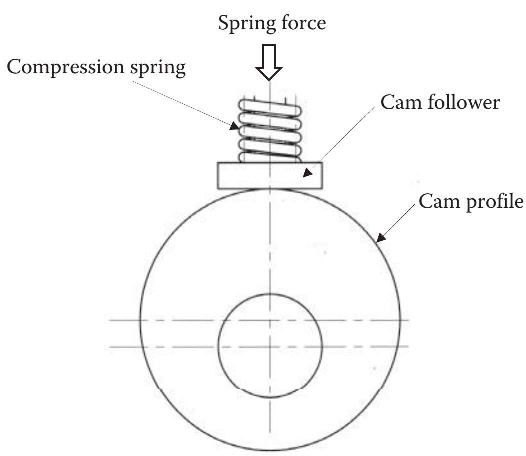


FIGURE 11.4 Open pairs.

11.5 PLANAR, SPHERICAL AND SPATIAL MECHANISMS

Linkages may be classified into three fundamental groups:

1. Planar
2. Spherical
3. Spatial

11.5.1 Planar Mechanism

A planar motion is defined as that, when all the particles that make up the body move in parallel planes. If the pivot points on a planar mechanism are projected, they will meet at infinity.

An example of a planar link is shown in Figure 11.5a. Planar links will be considered extensively in this introduction.

11.5.2 Spherical Mechanism

A body is considered to have spherical motion when all its particles move on the surface of concentric spheres. That is, if the pivot points are projected, they will all meet at the centre of a sphere. Figure 11.5b depicts a universal joint that moves in a spherical motion. Other examples include some types of robotic arms.

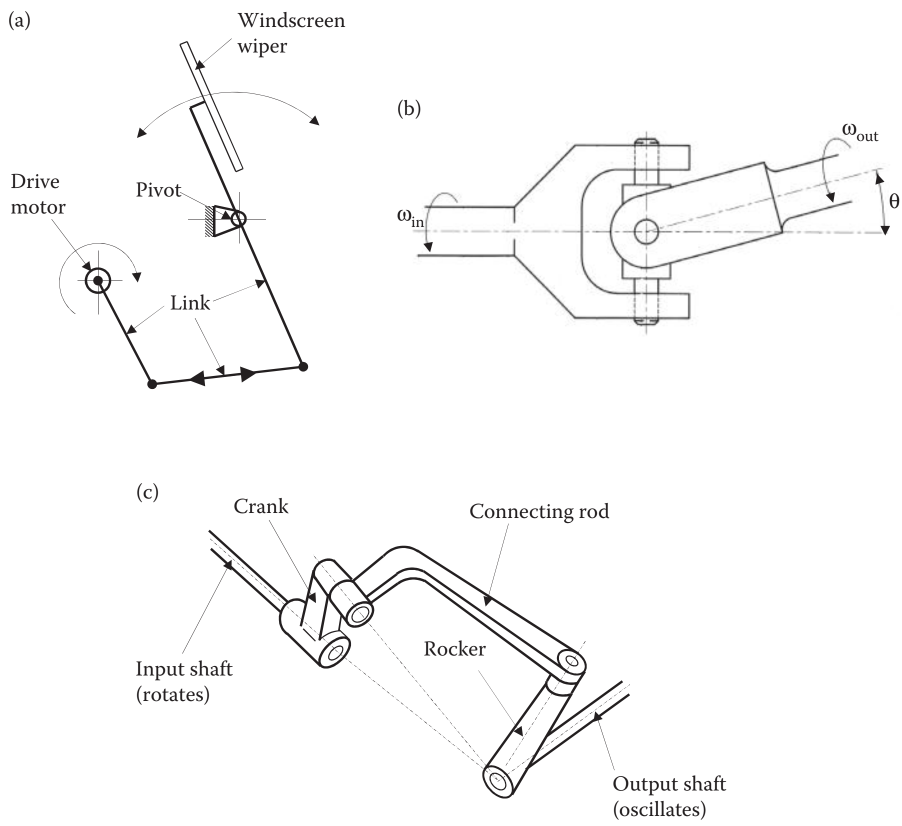


FIGURE 11.5 Planar, spherical and spatial mechanisms. (a) Example of a planar linkage, (b) example of a spherical mechanism (universal joint) and (c) example of a spatial mechanism.

11.5.3 Spatial Mechanism

In a spatial mechanism, particles that make up the body move in paths that do not always remain in a plane. Figure 11.5c illustrates one such spatial mechanism. The mechanism shown is referred to as a '*Stewart*' platform (Figure 11.6) and is used extensively as the base mechanism in flight simulator platforms. Other examples can also be seen in modern folding pushchair frames.

Both planar and spherical mechanisms can be considered as subsets of spatial mechanisms.

Mechanical linkages are used extensively in mechanical engineering where they are designed to transform a given input force and movement into a desired output force and movement.

The ratio of the output force to the input force is referred to as the mechanical advantage of the linkage. The ratio of the input velocity to the output velocity is known as the velocity ratio. The mechanical advantage and the velocity ratio are defined so that they yield the same number in an ideal linkage.

The definition of a link is that of a rigid body having two or more pairing elements that connect with other bodies for the specific purpose of transmitting force and/or motion.

In a linkage mechanism, at least one link occupies a fixed position and is referred to as either the ground or fixed link.

Linkages come in all sorts of arrangements covering a three-bar linkage to six-bar linkages, although they are not restricted to six bars. The more complicated linkages may have more, but these are outside the scope of this introduction. Stevenson and Watt linkages usually found on steam locomotives are generally six-bar linkages.

Fixed linkages that have only three bars and have no movement between adjacent links are usually called a truss which is used in structures.

One of the most commonly used linkages is that of the four-bar linkage (usually referred to as a four-bar) and is the simplest closed chain linkage. Four-bar consists of three moveable linkages connected to a fixed link and connected in a loop connected by four rotating joints. The joints are configured so that the links all move in parallel planes and the assemblage is called a 'planar four-bar linkage'.

Planar four-bar linkages are important mechanisms in machine design and examples include

- Double wishbone suspension
- Watt and Chebyshev linkages (these linkages give an approximately straight line motion)
- Crank slider (four-bar and one degree of freedom)
- Pantograph (four-bar and two DOF, i.e. only one pivot joint is fixed)

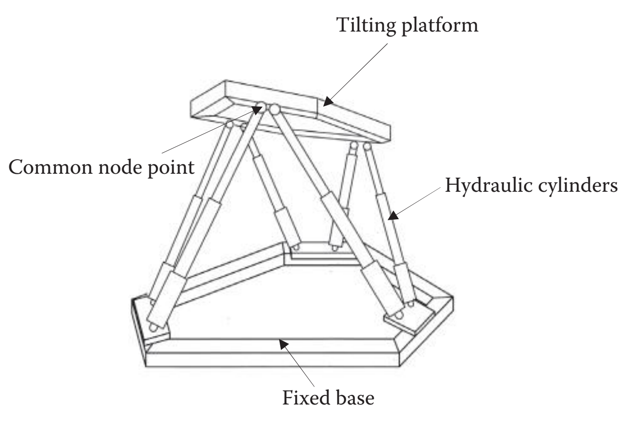


FIGURE 11.6 Stewart platform.

11.6 MOBILITY

When considering a linkage, an important consideration is the number of DOFs the mechanism will have. The mobility of a linkage is the number of input parameters which have to be controlled independently to bring the mechanism to a set position. This is, generally, the number of independent parameters including joint angles and slide distances that are needed to specify the configuration of the linkages disregarding any deformation of the links.

The mobility of a system is the sum of the unconstrained DOFs for the links in the system minus the constraints imposed by the joints.

A free link usually has three DOFs (x, y, θ). One link is always fixed and is referred to as either the fixed or ground link. Before any joints are attached, the number of DOFs of a linkage assembly with 'n' links is

$$DOF = 3(n - 1) \tag{11.2}$$

Connecting two links using a joint (which has one degree of freedom) adds two constraints. Connecting two links that have a joint with two DOFs includes one restraint to the system. The number of 1 DOF joint = j_1 and the number of joints with 2 DOF joints = j_2; the 'mobility' of a system can be expressed as

$$\text{Mobility} = m = 3(n - 1) - 2j_1 - j_2 \tag{11.3}$$

Figure 11.7 gives examples of the mobility of some planar linkages.

Note: In the case of a linkage that has a mobility of '0', this is essentially a structure.

This discussion will be developed further in Sections 11.7 and 11.8.

11.7 CHEBYSHEV–GRUBER–KUTZBACH CRITERION

In Section 11.6, the discussion introduced the concept of mobility and how it was arrived at. In this and the following sections, a more formalised method of establishing mobility is discussed.

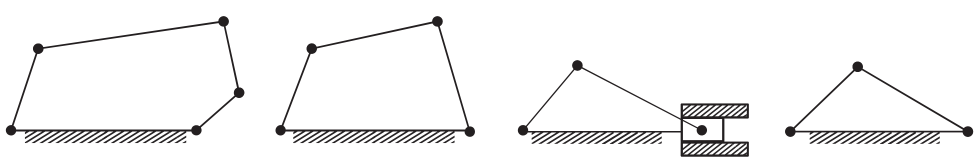
The Chebyshev–Gruber–Kutzbach criterion is used to determine the DOF within a kinematic chain of linkages, that is, a coupling of rigid bodies by means of mechanical constraints and it computes the number of parameters that define the configuration of a linkage that consists of a number of individual links and joints together with the DOF that exists at each joint.

Gruber's equation is written as

$$F = 3(n - 1) - 2I - h \tag{11.4}$$

where

F = total DOFs of the mechanism
n = number of links (including the frame)
I = number of lower pairs (one degree of freedom)
h = number of higher pairs (two DOFs)



m = 2, n = 5, j_1 = 5 and j_2 = 0 | m = 1, n = 4, j_1 = 4 and j_2 = 0 | m = 1, n = 4, j_1 = 4 and j_2 = 0 | m = 0, n = 3, j_1 = 3 and j_2 = 0

FIGURE 11.7 Mobility of planar linkages.

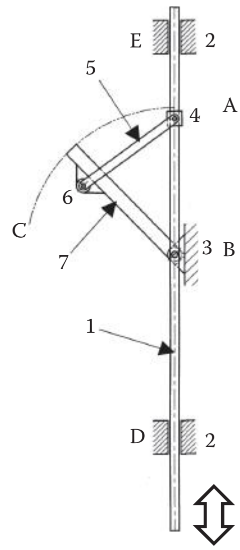


FIGURE 11.8 Example 11.1.

EXAMPLE 11.1

Consider a transom that is fitted above a door as shown in Figure 11.8 where the transom is opened and closed by means of the link (1).

It is required to calculate its degree of freedom.

Solution

n = 4 (link 1, 5, 6 and frame 2), l = 4 (at A, B, C and D), h = 0

$F = 3(4 - 1) - 2 \times 4 - 1 \times 0$

$F = 1$

Note: In this example, 'D' and 'E' as a prismatic pair count only as one lower pair.

The number of DOFs of a mechanism is also called the mobility of the mechanism. The mobility is the number of input parameters (usually paired variables) that have to be individually controlled to bring the mechanism into a particular desired position.

In order to control a mechanism, the number of independent input motions must equal the DOFs of the mechanism. As an example, the transom in Example 11.1 has only a single degree of freedom; therefore, only one independent input motion is required to open and close the transom, that is, push and pull on the link (1) opens and closes the window.

11.8 GRASHOF'S LAW

Prior to discussing Grashof's Law, it will be useful to extend the definitions as covered in Sections 11.3 and 11.4.

In a planar mechanism, the simplest group of lower pair mechanisms will be the four-bar chain or linkage and comprises four bar-shaped links and turning pairs as depicted in Figure 11.9a through e.

The fixed link is referred to as either the 'fixed link' or the 'frame'. The link that is opposite the frame is called the 'coupler link'. The links that are hinged to the frame are called 'side links'. A link that is free to rotate through 360° with respect to a second link is said to revolve relative to the second link (not necessary the frame). Where it is possible for all four links to become simultaneously aligned, such a state is called a 'change point'.

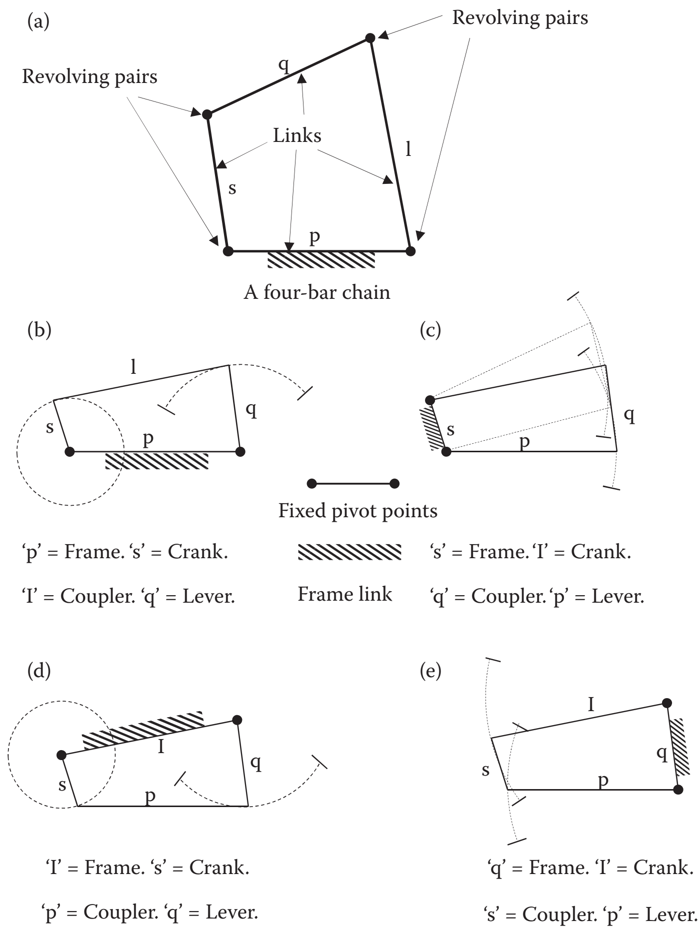


FIGURE 11.9 (a)–(e): Inversions of various linkages.

Some important concepts in link mechanisms are

1. *Crank:* A side frame that revolves relative to the frame is called a 'crank'.
2. *Rocker:* Any link that does not revolve is referred to as a 'rocker'.
3. *Crank–rocker mechanism:* In a four-bar chain, if the shorter side link revolves and the other one rocks (that is oscillates), this is called a 'double-crank' mechanism.
4. *Double-crank mechanism:* In a situation where both side links revolve, this is called a 'double-crank' mechanism.
5. *Double-rocker mechanism:* In a four-bar chain where both side links rock, this is called a 'double-rocker' mechanism.

11.8.1 Classification

Referring to Figure 11.9a, the following notation is defined:

In a four-bar chain, the line segment between hinges on a given link is called a 'bar' where 'b' is the length of the shortest bar, 'c' is the length of the longest bar and 'a' and 'd' are the lengths of intermediate bars.

In certain cases, it may be required to design a four-bar mechanism where the input link is driven and rotates continuously. It is essential that the input link is able to rotate through a complete revolution without any binding. Grashof provides a law which provides a simple test for this condition.

Grashof's law states that the sum of the longest and shortest links cannot be greater than the sum of the remaining links if there is to be continuous relative rotation between two members.

From Figure 11.9a through e, it can be seen that one of the links (usually the shortest link) will be able to rotate continuously if the following conditions are met

$$s(\text{shortest link}) + l(\text{longest link}) \leq p + q \tag{11.5}$$

This being so will satisfy Grashof's law and different inversions will be possible by fixing the different links within the chain.

The mechanism satisfying Grashof's criterion is called Grashof's linkage.

For a non-Grashof linkage, only a rocker–rocker mechanism will occur as in the case where

$$s(\text{shortest link}) + l(\text{longest link}) > p + q \tag{11.6}$$

In this case, all three mobile links will rock.

EXAMPLE 11.2

In a four-bar mechanism, the lengths of the driver-crank, coupler and follower link are 150, 250 and 300 mm, respectively. The fixed link length is 'L_a'. Determine the range of values for 'L_a' so as to make the mechanism:

1. Crank–crank mechanism
2. Crank–rocker mechanism

Solution

For the crank–crank mechanism, the conditions to be satisfied are

$$150\text{ mm} + 250\text{ mm} \leq L_a + 300\text{ mm}$$

$$400\text{ mm} \leq L_a + 300\text{ mm}$$

$$L_a \leq 100\text{ mm} \qquad \text{(Ans)}$$

For the crank–rocker mechanism, the conditions to be satisfied are

- The link adjacent to the fixed link must be the smallest link.
- $s + l \leq p + q$.

Two possibilities have to be considered:

1. When L_o is the longest link.
2. When L_o is not the longest link.

When L_o is the longest link, from Grashof's criterion:

$$L_o + 150 \leq 250 + 300$$

or

$$L_o \leq 400$$

When L_o is not the longest link, from Grashof's criterion:

$$300 + 150 \le L_o + 250$$

or

$$L_o \ge 200$$

Hence, the crank–rocker mechanism, range of values for L_o is

$$200 \le L_o \le 400 \text{ mm (Ans)}$$

11.9 FOUR-BAR LINKAGE

11.9.1 Planar Four-Bar Linkages

The four-bar linkage or four-bar chain is fundamental in a wide range of mechanisms and takes many different shapes and sizes. As a mechanism, it is fairly simple to manufacture, inexpensive, easy to maintain and robust. It has the ability to operate at low and high speeds, transmit small or large loads and operate in a wide range of environments with suitable protection. It is possible to design planar four-bar linkages to cover a wide range of movements including complex path patterns. Examples of planar four-bar linkages can be found in a wide range of industries including earth-moving machines, landing gear systems in aviation, suspension units in the automotive industry and automatic assembly machines in the manufacturing industry, to name but a few.

11.9.2 Inversion

A mechanism is defined as a kinematic chain where one of the links is fixed; thus, by fixing the individual links one at a time, a number of variants of the mechanisms are obtained. These variants are dependent upon the number of links. These variants of the kinematic chain are called 'inversions of kinematic chain' or 'inversions of mechanism'.

It should be noted that the relative motion between the various links is not changed in any manner through the process of inversion, but their absolute motions (measured in respect of the fixed link) may be altered drastically.

Most mechanisms are reversible where in one configuration the 'driver' link transmits motion to a 'follower'; in another configuration of the same mechanism, the role is reversed and the driver link then becomes the follower and the link that initially was the follower then becomes the driver. An example of this is when a reciprocating steam engine is considered where the piston is the driver and the flywheel is the follower, while in a reciprocating air compressor, the flywheel becomes the driver.

Consider the four-bar linkage depicted in Figure 11.10a. Figure 11.10b through e shows the various inversions possible with the basic linkage. The four inversions look identical to each other but their mobility can be significantly varied by altering the proportions of lengths of the various links.

Planar four-bar linkages are constructed from links connected in a loop by four one degree of freedom joints. A joint may be either hinged (revolute), denoted by 'R', or sliding (prismatic), denoted by 'P'. A planar quadrilateral linkage formed of four links and four revolute joints is designated 'RRRR'. A slider–crank linkage constructed from four links and connected by three revolute joints and one prismatic joint is designated 'RRRP'. If a double slider is used, this will be a 'PRRP' linkage (see Figure 11.11).

As described earlier in the chapter, one link of the chain is usually fixed and is called the 'ground' link, fixed link or the frame. The two links that are connected to the frame are referred to as the 'grounded links' and these are usually the input and output links of the system. The remaining link

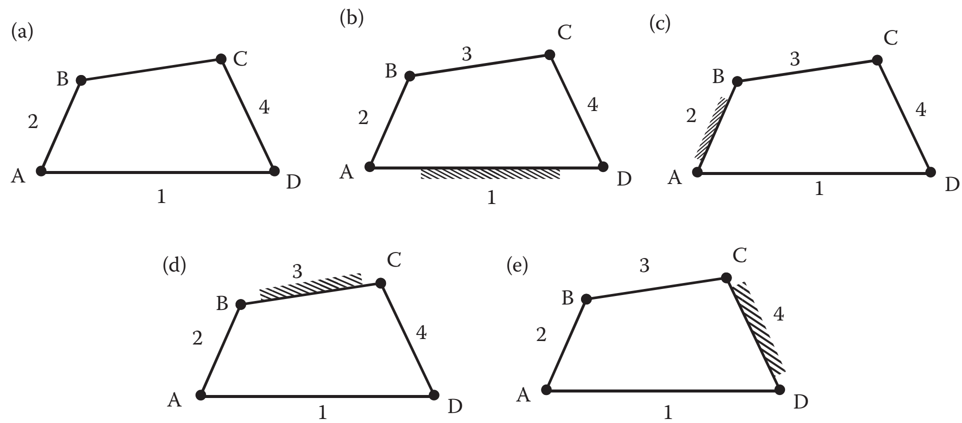


FIGURE 11.10 Variants of a four-bar chain. (a) Basic kinematic chain, (b) variant 'b', (c) variant 'c', (d) variant 'd' and (e) variant 'e'.

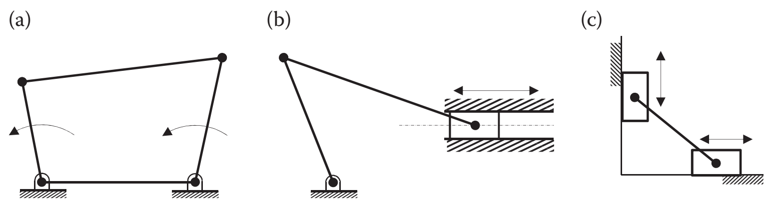


FIGURE 11.11 Examples of revolute and prismatic joints. (a) Planar linkage (RRRR), (b) slider-crank linkage (RRRP) and (c) double-slider linkage (PRRP).

is called the 'floating link' and this is also called a 'coupler' or 'connecting link' as it connects the input to the output.

Section 11.8 describes Grashof's law for a planar quadrilateral four-bar linkage where the sum of the shortest link + longest link should be either shorter or equal to the sum of the remaining links. This is the requirement for the shortest link (usually referred to as the crank) to rotate fully with respect to a neighbouring link.

11.9.3 Slider–Crank Linkage

A slider–crank linkage or chain is a specialised form of a four-bar linkage where one turning pair is replaced by a sliding pair. Various inversions of slider–crank linkages are obtained by fixing different links and these are discussed further below with reference to Figure 11.12.

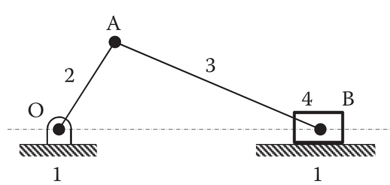


FIGURE 11.12 Slider–crank mechanism.

11.9.3.1 Link 1 Fixed

If link 1 of Figure 11.12 is fixed and link 2 is allowed to act as a crank, the resulting mechanism is called a slider–crank mechanism. This converts reciprocating motion into rotary motion or vice versa.

11.9.3.2 Link 2 Fixed

When link 2 of the slider–crank linkage is fixed and link 3 acts as a crank, link 1 will rotate about centre ‘O’ along the slider 4, which in turn will reciprocate along the link as in Figure 11.13. This inversion produces two popular forms of mechanisms:

1. Whitworth quick return mechanism
2. Rotary engine configuration

Figure 11.14 depicts the Whitworth quick return mechanism that is used in a number of machine tools including the ‘shaper’ in which a fixed tool is passed over a work piece at the correct cutting speed and returned to its starting point in a shorter time, thereby reducing the overall cycle time.

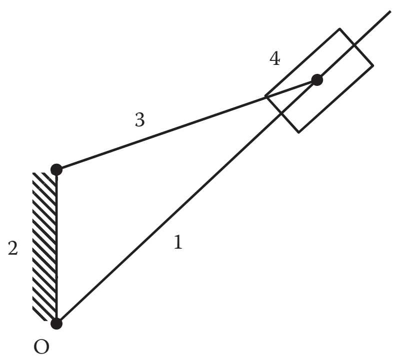


FIGURE 11.13 Inversion of a slider–crank chain.

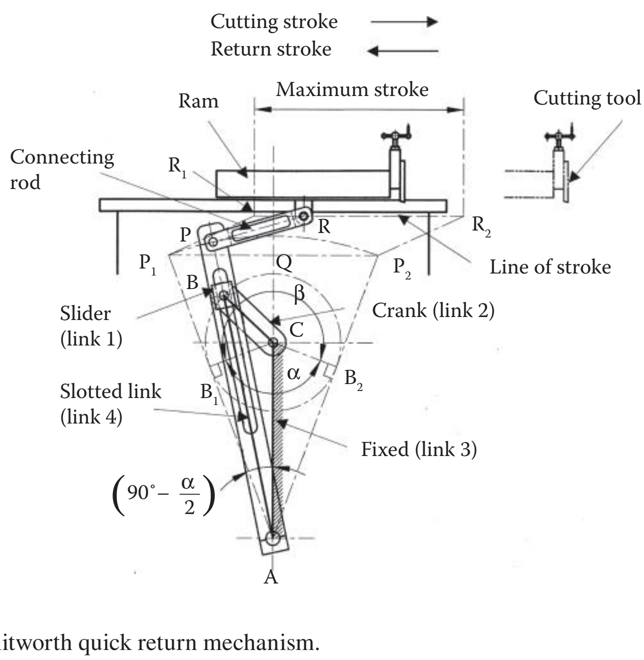


FIGURE 11.14 Whitworth quick return mechanism.

In this mechanism, the link 'A:C' (link 3) that forms the turning pair is fixed. The driving crank 'C:B' revolves at a constant angular speed around the centre 'C'. A sliding block which is attached to the crank pin at 'B' is fitted in and slides in the slotted bar 'A:P'. This causes the slotted bar to oscillate about the pivot point 'A'. The slotted link is connected to the shaper ram that carries the cutting tool via a connecting link 'P:R' and this reciprocates along the line of stroke 'R_1:R_2'.

Positions 'A:P_1' and 'A:P_2' are tangential to the circle 'A:B_1' and 'A:B_2'. The forward-cutting action occurs when the crank rotates from the position 'C:B_1' to 'C:B_2' (or through the angle 'β') when rotating in the clockwise direction. The return stroke occurs when the crank rotates from position 'C:B_2' to 'C:B_1' (or through the angle 'α' when rotating in the clockwise direction).

As the crank is rotating at a uniform angular speed, it follows that

$$\frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{\beta}{\alpha} = \frac{\beta}{360° - \beta} \tag{11.7}$$

or

$$\frac{360° - \alpha}{\alpha} \tag{11.8}$$

As the cutting tool travels the distance of 'R_1:R_2' during the total cutting cycle, the length of stroke will be

$$= R_1R_2 = P_1P_2 = 2 \cdot P_1Q = 2P_1 \sin\angle P_1 \cdot AQ \tag{11.9}$$

$$= 2AP_1 \sin\left(90° - \frac{\alpha}{2}\right) = 2AP\cos\frac{\alpha}{2} \quad \text{where } AP_1 = AP \tag{11.10}$$

$$2AP \times \frac{CB_1}{AC} \qquad \text{where } \cos\frac{\alpha}{2} = \frac{CB_1}{AC} \tag{11.11}$$

$$2AP \times \frac{CB}{AC} \qquad \text{where } CB_1 = CB \tag{11.12}$$

From a study of Figure 11.14, it will be noted that the angle 'β' which produces the forward stroke of the ram is larger than the angle 'α' which produces the return stroke. As the crank is rotating at a uniform angular rotation, the return stroke will be completed in a shorter time to that of the forward stroke. For this reason, this mechanism is known as a quick return mechanism.

The output from a planar four-bar linkage is not restricted to the output link; in some applications, the output of the four-bar linkage could be taken from the coupler. In Figure 11.9 where the motion is required to be linear, such an application may be an automatic assembly machine such as a 'pick and place'. This is also an example of a double-rocker mechanism.

Extensive use is made of loci generated by coupler curves in a wide range of industries from automatic assembly machines as described to industrial manipulators.

There are a number of 'atlases' containing a wide range of curves generated by points on the coupler link or extensions of the coupler in four-bar linkages. One such atlas has been produced by Hrones and Nelson which covers thousands of curves, but these are now being overtaken by modern simulation packages such as MATLAB® reducing the number of iterations to arrive at a more optimised curve.

11.10 MECHANICAL ADVANTAGE OF A FOUR-BAR LINKAGE

One of the principal criteria for the designer to be aware of is the ability of a particular mechanism to be able to transmit force or torque. In some mechanisms including gear trains, these mechanisms transmit a constant torque ratio between the input and the output. In a linkage, this is not the case, as the geometric nature of the linkage will be varying all the time through the mechanism's motion, and as such there will be a variation between the input and output ratios.

If it is considered that the mechanism is a conservative system (i.e. energy losses due to friction, heat sound, etc. are negligible when compared with the total energy that is transmitted by the system as shown in Figure 11.15), it also assumes that the effects of inertial forces are also negligible, that is, 'power in' will equal 'power out' ($P_{in} = P_{out}$).

Hence, the 'torque in' (T_{in}) times the 'angular velocity in' (ω_{in}) will equal the 'torque out' (T_{out}) times the 'angular velocity out' (ω_{out}).

That is

$$P_{in} = T_{in} \cdot \omega_{in} = T_{out} \cdot \omega_{out} = P_{out} \tag{11.13}$$

Alternatively:

$$P_{in} = F_{in} \cdot V_{in} = F_{out} \cdot V_{out} = P_{out} \tag{11.14}$$

where T_{in} and F_{in} are torques applied to the linkage chain and T_{out} and F_{out} are those that are exerted by the chain, and V_{in} and V_{out} are the velocities of the points through which F_{in} and F_{out}, respectively, act.

In vector form:

$$V \cdot F = VF \cos(\arg F - \arg V) \tag{11.15}$$

Also,

$$V \cdot F = V_x F_x + V_y F_y \tag{11.16}$$

It should be noticed that the units of torque times angular velocity together with the scalar products of force times velocity both represent power as in Equation 11.13.

That is

$$\frac{T_{out}}{T_{in}} = \frac{\omega_{in}}{\omega_{out}} \tag{11.17}$$

These relationships are dimensionless.

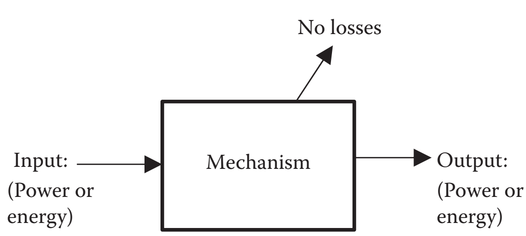


FIGURE 11.15 Conservation of power and energy through a mechanism.

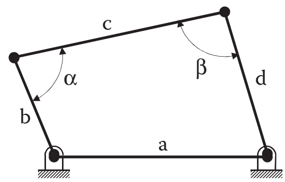


FIGURE 11.16 Mechanical advantage.

In cases where the input force generates an output torque (or vice versa):

By definition, the 'mechanical advantage (MA) of a system is the ratio of the magnitude of 'F_{out}' over 'F_{in}', that is

$$MA = \frac{F_{out}}{F_{in}} \tag{11.18}$$

where $F = |F|$

Torque is the product of force 'F' times radius 'r'. Combining Equations 11.13 and 11.18 and solving for MA, the following equation will result

$$MA = \left(\frac{T_{out}}{r_{out}}\right)\left(\frac{r_{in}}{T_{in}}\right) = \left(\frac{r_{in}}{r_{out}}\right)\left(\frac{T_{out}}{T_{in}}\right) \tag{11.19}$$

and

$$MA = \left(\frac{r_{in}}{r_{out}}\right)\left(\frac{\omega_{in}}{\omega_{out}}\right) \tag{11.20}$$

In SI units, these dimensions are rad/s to m/s and N/Nm (or the inverse).

A power, force or torque requirement can be converted into a velocity ratio and velocity analysis can then be used to design a mechanism for a required mechanical advantage.

Bear in mind that velocity ratio is normally a function of the mechanism position, and therefore the mechanical advantage will also be a function of the mechanism position.

To summarise, the mechanical advantage of a planar four-bar linkage is defined as the ratio of the output power or torque exerted by the driven link or follower link to the required input power or torque at the driver link.

The mechanical advantage of the four-bar linkage is directly proportional to the sine of the angle 'γ' between the coupler and the follower and inversely proportional to the sine of the angle 'β' between the coupler and the driver. For one complete cycle of the mechanism, both these angles 'γ' and 'β' are continuously changing and as a result the mechanical advantage also changes as the linkage moves (see Figure 11.16).

11.11 FREUDENSTEIN'S EQUATION

Freudenstein developed a simple algebraic method for determining the position of an output link knowing the length of the four links and the position of the input link.

Consider the four-bar linkage chain shown in Figure 11.17.

The position vector of the links has the following relationship:

$$l_1 + l_2 + l_3 + l_4 = 0 \tag{11.21}$$

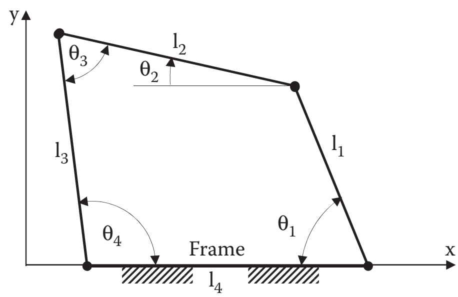


FIGURE 11.17 Four-bar linkage vector representation.

Equating the horizontal distances:

$$l_1 \cos \theta_1 + l_2 \cos \theta_2 + l_3 \cos \theta_3 + l_4 \cos \theta_4 = 0 \tag{11.22}$$

Equating the vertical distances:

$$l_1 \sin \theta_1 + l_2 \sin \theta_2 + l_3 \sin \theta_3 + l_4 \sin \theta_4 = 0 \tag{11.23}$$

Making the assumption that $\theta_1 = 180°$, hence, $\sin \theta_1 = 0$, $\cos \theta_1 = -1$.
Rearranging and squaring both sides in respect to l_3:

$$l_3^2 \cos^2 \theta_3 = (l_1 - l_2 \cos\theta_2 - l_4 \cos\theta_4)^2 \tag{11.24}$$

$$l_3^2 \sin^2 \theta_3 = (-l_2 \sin \theta_2 - l_4 \sin \theta_4)^2 \tag{11.25}$$

Summing Equations 11.24 and 11.25 and using the following relationships:

$$\cos(\theta_2 - \theta_4) = \cos\theta_2 \cos\theta_4 + \sin\theta_2 \sin\theta_4 \text{ and } \sin^2 \theta + \cos^2 \theta = 1 \tag{11.26}$$

The following relationship will result:

$$\frac{l_3^2 - l_1^2 - l_2^2 - l_4^2}{2 \cdot l_2 - l_4} + \frac{l_1}{l_4} \cos \theta_2 + \frac{l_1}{l_2} \cos \theta_4 = \cos(\theta_2 - \theta_4) \tag{11.27}$$

The following will result in Freudenstein's equation:

$$K_1 \cos\theta_2 + K_2 \cos\theta_4 + K_3 = \cos(\theta_2 - \theta_4) \tag{11.28}$$

where

$$K_1 = \frac{l_1}{l_4} \tag{11.29}$$

$$K_2 = \frac{l_1}{l_2} \tag{11.30}$$

$$K_3 = \frac{(l_3^2 - l_1^2 - l_2^2 - l_4^2)}{2 \cdot l_2 \cdot l_4} \tag{11.31}$$

The equation can be used to design a mechanism from a required relationship between input to output; in this case, the unknowns are the lengths of the links or more frequently the ratios of the links l_2/l_1, l_4/l_1 and l_3/l_1. If three desired values of the output are fixed for three desired values of the input (co-ordinating input and output positions), three equations in three unknowns are produced and hence the ratio of the links relating to the length of the fixed link l_1.

To obtain an expression for the output θ_4, it will be found more convenient to write Equation 11.28 as follows:

$$K_1 \cos\theta_2 + K_2 \cos\theta_1 - K_3 = \cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1 \quad (11.32)$$

Collecting terms containing θ_1 gives

$$\sin\theta_1 \sin\theta_2 + \cos\theta_1 (\cos\theta_2 - K_2) = K_1 \cos\theta_2 - K_3$$
$$A \sin\theta_1 + B \cos\theta_1 = C \quad (11.33)$$

$$\left.\begin{aligned} A &= \sin\theta_2 \\ B &= (\cos\theta_2 - K_2) \\ C &= (K_1 \cos\theta_2 - K_3) \end{aligned}\right\}$$

Solving for l_1, recall from trigonometry that

$$\sin\theta_1 = \frac{2t}{1+t^2}$$

where

$$t = \tan\frac{1}{2}\theta_1$$

and

$$\cos\theta_1 = \frac{1-t^2}{1+t^2}$$

Equation 11.33 becomes

$$(B + C)t^2 - 2At + (C - B) = 0 \quad (11.34)$$

This equation is a quadratic in 't' and the solution will be

$$t = \frac{A \pm \sqrt{A^2 + B^2 - C^2}}{B + C} \quad (11.35)$$

There being two solutions:

$$\theta_1^+ = 2 \tan^{-1} \frac{A + \sqrt{A^2 + B^2 - C^2}}{B + C} \quad (11.36)$$

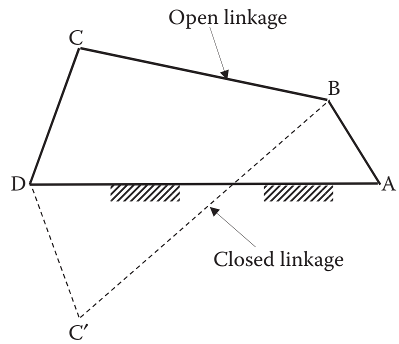


FIGURE 11.18 Open and closed linkages.

Corresponding to the positive sign, and

$$\theta_1^- = 2\tan^{-1}\frac{A - \sqrt{A^2 + B^2 - C^2}}{B + C} \tag{11.37}$$

These two values relate to the two possible ways that the linkage can be assembled as depicted in Figure 11.18 for the given values of l_1, l_2, l_3, l_4 and ABCD will be referred to as an 'open' linkage and ABCD as a 'crossed' linkage.

EXAMPLE 11.3

A four-bar linkage has the following parameters:

Crank	= 100 mm
Coupler	= 200 mm
Follower	= 200 mm
Fixed link	= 200 mm

Consider the quadrilateral chain shown in Figure 11.19. Determine the Freudenstein equation when the input link (l_1) is indexed every 15° between 0° and 360°.

The Freudenstein equation is given as

$$K_1 - K_2 \cdot \cos\theta_4 + K_3 - \cos(\theta_4 - \theta_1) = 0$$

where

$$K_1 = \frac{l_4}{l_3} = \frac{200}{100} = 1.00$$

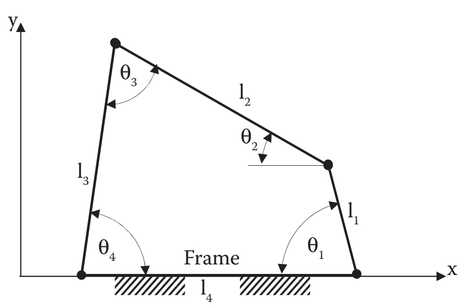


FIGURE 11.19 Example 11.3, kinematic chain.

$$K_2 = \frac{l_4}{l_1}$$

$$K_3 = \frac{(l_1^2 - l_2^2 + l_3^2 + l_4^2)}{2 \cdot l_1 \cdot l_3}$$

Table 11.1 tabulates the results from the above equations.

TABLE 11.1
Example of Freudenstein's Equation, Four-Bar Linkage—Freudenstein's Equation

Link Lengths (mm)

a	100
b	200
c	200
d	200
Ratio 1	1
Ratio 2	2
Ratio 3	1.25

Input Angle, θ_1		Output Angle, θ_2		
Degree	**Radian**	**Degree**	**Radian**	**Freudenstein's Equation**
0	0.0000	41	0.7156	5.2120
15	0.2618	47	0.8203	3.4004
30	0.5236	53	0.9250	4.6194
45	0.7854	60	1.0472	4.9145
60	1.0472	68	1.1868	1.5152
75	1.3090	75	1.3090	–0.5935
90	1.5708	83	1.4486	0.9970
105	1.8326	89	1.5533	2.1873
120	2.0944	94	1.6406	–0.3358
135	2.3562	97	1.6930	3.1452
150	2.6180	96	1.6755	3.4402
165	2.8798	89	1.5533	0.4053
180	3.1416	76	1.3264	1.5482
195	3.4034	60	1.0472	5.1509
210	3.6652	48	0.8378	3.3238
225	3.9270	40	0.6981	4.5219
240	4.1888	34	0.5934	3.7234
255	4.4506	31	0.5411	1.0047
270	4.7124	29	0.5061	4.3657
285	4.9742	29	0.5061	37.859
300	5.2360	29	0.5061	30.661
315	5.4978	31	0.5411	0.1115
330	5.7596	33	0.5760	2.3958
345	6.0214	37	0.6458	–0.2732
360	6.2832	41	0.7156	4.0967

Note: Freudenstein's equation: 'F' = Ratio 1 – Ratio 2 × cos θ_4 + Ratio 3 – cos $(\theta_1 - \theta_4) = 0$.

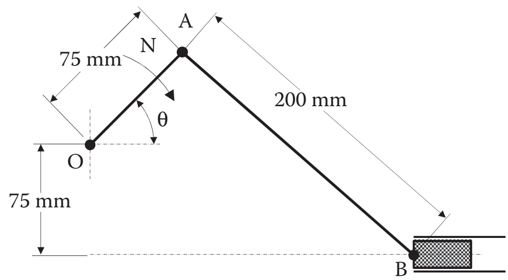


FIGURE 11.20 Space diagram for a slider–crank mechanism.

11.12 DRAWING VELOCITY VECTORS FOR LINKAGES

In the slider–crank linkage example shown in Figure 11.20, link ‘OA’ is driven and connects to a further link ‘AB’. This, in turn, connects to a slider block at ‘B’, which is allowed to reciprocate in a horizontal plane.

This section will concentrate on determining the velocity of the links using a graphical method for evaluating the link velocities. This is considered easier than using vector arithmetic and is capable of achieving accurate values.

EXAMPLE 11.4

To demonstrate the method, again considering Figure 11.20, the following values are applied to the figure. The individual link velocities will be determined for the position shown:

‘OA’ = 75 mm
‘AB’ = 200 mm
‘x’ = 75 mm
‘θ’ = 45°
‘N’ = 2000 rev/min

In this example, there are eight steps required to find the absolute velocity of the slider ‘B’ as well as the relative velocity of the slider with respect to the joint ‘A’:

Step 1: Draw a space diagram of the mechanism. This will generally be a scaled representation of the mechanism at the desired orientation of the links. The scale chosen will be determined on the method of drawing. If the drawing is being sketched, the scale will be dictated by the size of the paper being used. Alternatively, if the sketch is being drawn using a suitable CAD type program, then the sizes will be the actual dimensions.

Step 2: The first vector ‘OA’ is drawn as in Figure 11.21a with a length of 75 mm inclined at an angle of 45° to the horizontal.

Step 3: The diagram is continued by drawing in the vector ‘AB’ having a length of 200 mm and intersecting with the axis, which is 75 mm below the fixed point ‘O’ at ‘B’. At the completion of step 3, a diagram will consist of the two lines ‘OA’ and ‘AB’ as shown in Figure 11.21b.

Step 4: Calculation of the linear velocity of joint ‘A’.

The rotational speed ‘N’ of the link ‘OA’ = 2000 rev/min; this is equivalent to 209.44 rad/s.

The linear velocity of the link ‘OA’ = angular velocity × length of the link ‘OA’

$$\text{Linear velocity of link OA} = 209.44\ \text{rad/s} \times 75\ \text{mm} = 15{,}708\ \text{mm/s}$$

Step 5: The velocity vector diagram can now be progressed by drawing the velocity vector for the link ‘OA’. This is drawn perpendicular to the link having a length equivalent to

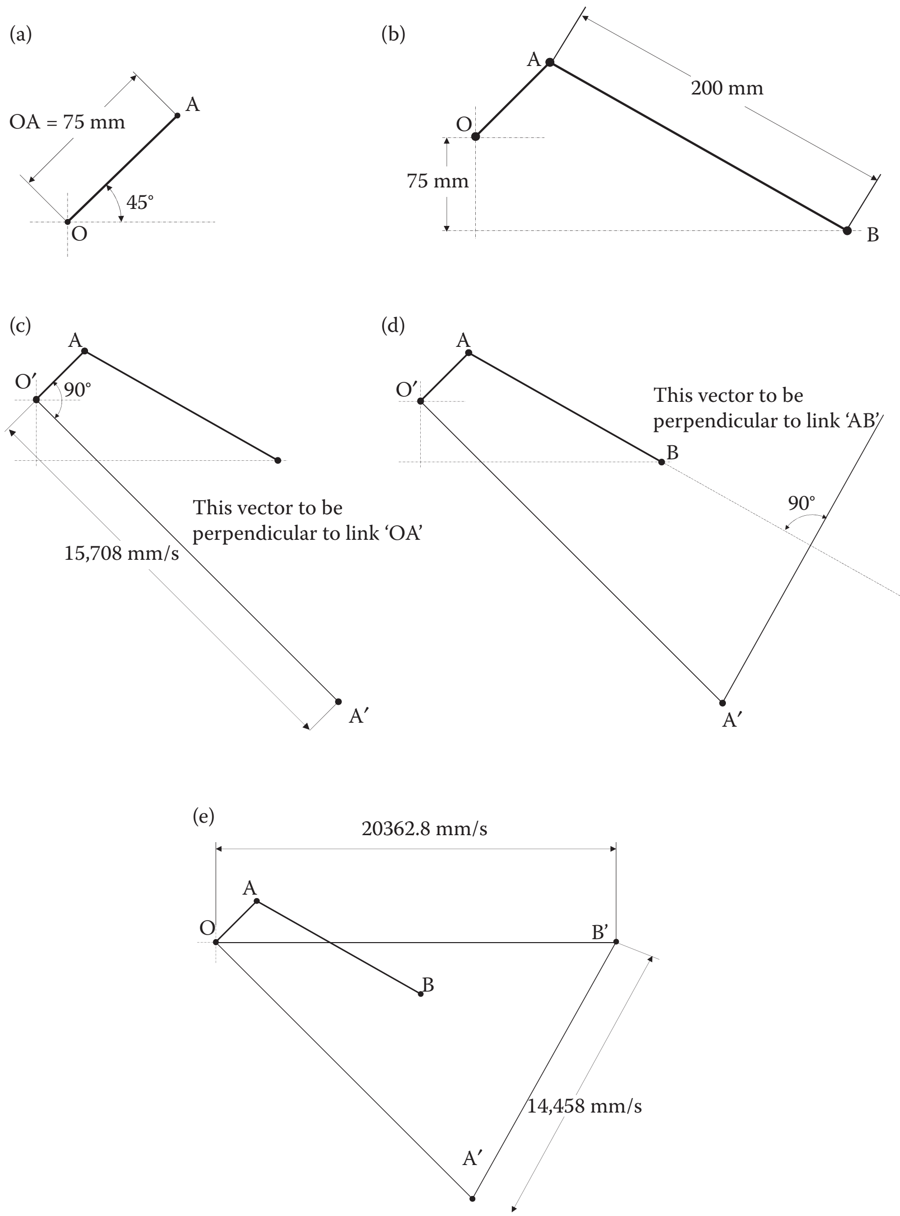


FIGURE 11.21 Velocity diagrams for Example 11.4. (a) Link OA step 2, (b) space diagram at step 3, (c) velocity diagram at step 5, (d) velocity diagram at step 6 and (e) velocity diagram at step 7.

15,708 mm, depending on the scale being drawn. This will represent the velocity vector $V_{oa'}$ with a value of 15,708 mm/s as shown in Figure 11.21c.

Step 6: The following step is to draw the velocity vector for the link 'AB'. There are two points known about this vector:

- It will start from the point 'A_1' of the velocity vector diagram.
- It will be perpendicular to the orientation of the link 'AB' as shown in Figure 11.20.

At this point, the value of the 'A_1B_1' is not known (the linear velocity of the joint 'B' with respect to the joint 'A'). The diagram will look similar to Figure 11.21d.

Step 7: To find the location of point 'B_1'', the use of the property of point 'B' will be used. Point 'B' will always have the motion only in the horizontal direction; hence, it will pass through the point 'O' (Joint 'B' is connected to the slider and is only allowed to reciprocate in the horizontal direction). Connecting the horizontal line with point 'O' gives the velocity of the joint 'B' with respect to joint 'O'. This now completes the final velocity vector of the mechanism. The final velocity diagram will look like Figure 11.21e.

Step 8: All that remains is to measure the length of the vectors to obtain the velocity of joint 'B' with respect to 'A' from the length of the line 'A_1B_1' which measures 14,458 mm/s and the velocity of point 'B' with respect to 'O' (or the ground) measuring 20,362.8 mm/s.

This procedure can be repeated for the mechanism with varying values of 'θ' and the velocity profile of the mechanism can be resolved.

The method outlined above can be used to determine velocity vectors for other mechanisms without difficulty. Its prime advantage is that the method is easily understood.

Section 11.13 will focus on drawing the acceleration vector diagram.

11.13 DRAWING ACCELERATION VECTORS FOR LINKAGES

In Section 11.12, a velocity diagram for a slider–crank mechanism was developed. In this section, an acceleration vector diagram for the same example will be discussed.

Reiterating, the space diagram for the slider–crank mechanism is shown in Figure 11.20.

Before proceeding, there are a number of points that need to be borne in mind regarding the acceleration diagrams for a mechanism.

Item 1: Typically, there will be two components of acceleration:

- Radial
- Tangential

Item 2: The direction of the radial component of acceleration will always be parallel to the orientation of the link.

Item 3: The direction of the tangential component will always be perpendicular to the orientation of the link.

Item 4: In the case of the end of a link that is moving at a constant angular velocity (i.e., a link that is pivoting at one end and rotating at a constant rev/min with respect to the pivot point), the rotating end will only have a radial acceleration component with no tangential component.

Item 5: Where a joint moves in a straight line (such as a slide or a cylinder), there will be no radial acceleration component. The direction of the total acceleration of the joint, in such cases, will be parallel to the line of the movement of the joint.

Item 6: The radial acceleration for a link can be calculated from the following equation:

$$A_{xy} = xy \times W_{xy}^2 = \frac{V_{xy}^2}{xy}$$

where

A_{xy} is the radial acceleration of joint 'x' with respect to joint 'y'
W_{xy} is the angular velocity of joint 'x' with respect to joint 'y'
V_{xy} is the linear velocity of joint 'x' with respect to joint 'y'
xy is the length of link 'xy'

EXAMPLE 11.5

The following steps are required to draw an acceleration vector diagram for the slider–crank mechanism shown in Figure 11.20.

Step 1: Orientation of link 'OA'. The joint 'A' is rotating at a constant angular velocity (2000 rev/min) about joint 'O'; from item 4, joint 'A' will only have a radial component with no tangential component of acceleration. From item 2, the radial component in the acceleration vector diagram will be orientated parallel to the link 'OA'.

Step 2: Value of the radial acceleration component for the link 'OA'. From the Example statement 11.12, the angular velocity of the link 'OA' (Woa) is 2000 rev/min = 209.44 rad/s and the length of the link 'OA' is 75 mm.

From Item 6, the radial acceleration component for the link 'OA' is calculated as

$$\text{Aroa} = 75\,\text{mm} \times 209.44^2 = 3{,}289{,}884\,\text{mm/s}^2$$

Step 3: Representing Aroa in the acceleration diagram. Due to the very large numbers encountered with the acceleration values, it will be sensible to give them a scale factor to make the drawing more manageable. In this instance, a scale factor of 10^4 will give the length of the vector Aroa of 328.99 mm in the acceleration vector diagram. As there is no tangential component for the link, 'Arao' will represent Aao; this is shown by the vector 'O_2A_2' in Figure 11.22a.

Step 4: Drawing the radial component of acceleration for the joint 'B' with respect to 'A', Arba for the link 'AB'.

Start by drawing a line from point 'A_2' to the point 'z' as shown in Figure 11.22b and orientated along the link 'AB' of the space diagram.

Using item 6 to calculate the value of Arba:

$$\text{Arba} = \frac{(\text{Vba})^2}{\text{BA}}$$

$$= \frac{14{,}458^2}{200}$$

$$= 1{,}045{,}168.82\ \text{mm/s}^2$$

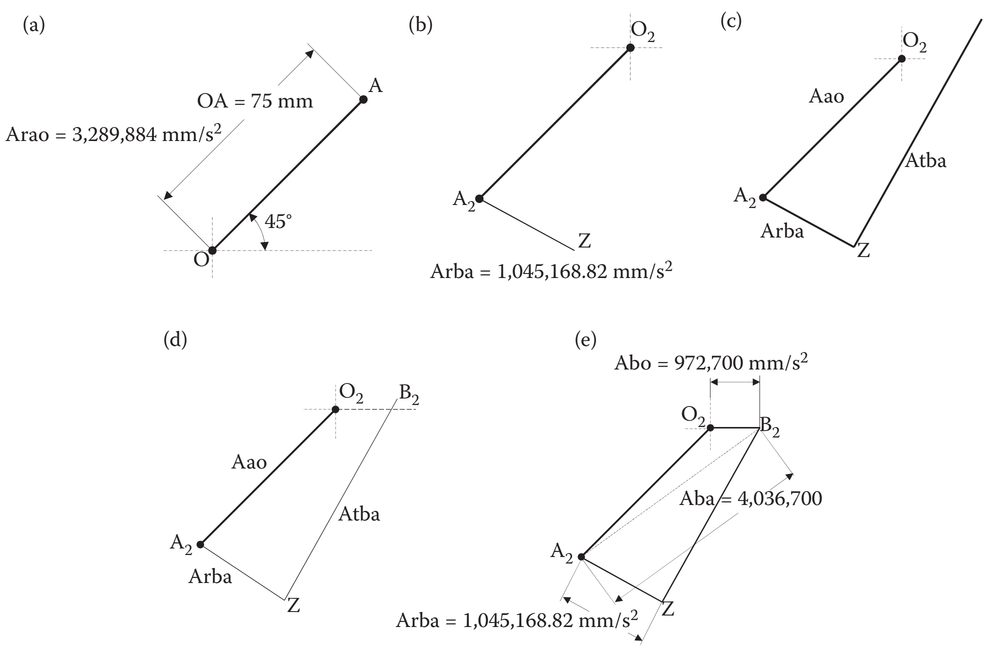


FIGURE 11.22 Acceleration diagrams for Example 11.5. (a) Link OA step 1, (b and c) acceleration diagram Step 5 and (d and e) acceleration diagram Step 6.

The values of 'Vba' and 'BA' are obtained from Example 11.4.

The scaled value of Arba = 1,045,168.82/10^4 = 104.517 mm.

Step 5: Drawing the tangential component (Atba) for the link AB. From item 3, the tangential component of the acceleration will be perpendicular to the radial component. At the moment, the value of the tangential component (Atba) of acceleration is unknown.

Referring to Figure 11.22c, a line is drawn perpendicular to the line A_2z.

Step 6: Drawing the acceleration vector for the slider 'B'. From item 5, the total acceleration of the joint 'B' with respect to the joint 'O' is represented by a horizontal straight line starting from 'O_2' until it intersects the line representing the vector 'Atba'.

In Figure 11.22d, the line O_2 represents the total linear acceleration for the slider with respect to joint 'O'.

Step 7: Calculating the output acceleration values. The length of the lines 'A_2B_2' and 'O_2B_2' represents the linear acceleration of the joint 'B' with respect to the joint 'O'. This is shown in Figure 11.22e.

Measuring the respective lengths and multiplying by the scale factor, the following values are obtained:

Linear acceleration of joint 'B' with respect to joint 'A'

$$\text{Aba} = 4{,}036{,}700\ \text{mm/s}^2$$

Linear acceleration of joint 'B' with respect to the joint 'O'

$$\text{Abo} = 972{,}700\ \text{mm/s}^2$$

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